LESSON 1-3

Distance and Midpoints

Then

- You graphed points on the coordinate plane.

Now

1. Find the distance between two points.
2. Find the midpoint of a segment.

Why?

- The location of a city on a map is given in degrees of latitude and longitude. For short distances, the Pythagorean Theorem can be used to approximate the distance between two locations.

NewVocabulary

distance
irrational number
midpoint
segment bisector

Common Core State Standards

Content Standards

G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Mathematical Practices

2 Reason abstractly and quantitatively.

7 Look for and make use of structure.

1 Distance Between Two Points

The **distance** between two points is the length of the segment with those points as its endpoints. The coordinates of the points can be used to find this length. Because $\overline{PQ}$ is the same as $\overline{QP}$, the order in which you name the endpoints is not important when calculating distance.

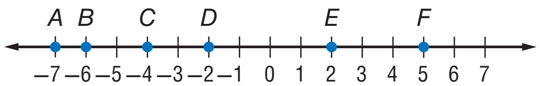
KeyConcept Distance Formula (on Number Line)

Words The distance between two points is the absolute value of the difference between their coordinates.

Symbols If P has coordinate x_1 and Q has coordinate x_2, $PQ = |x_2 - x_1|$ or $|x_1 - x_2|$.

Example 1 Find Distance on a Number Line

Use the number line to find BE.



The coordinates of B and E are −6 and 2.

$BE = |x_2 - x_1|$ Distance Formula

$= |2 - (-6)|$ $x_1 = -6$ and $x_2 = 2$

$= 8$ Simplify.

GuidedPractice

Use the number line above to find each measure.

1A. AC **1B.** CF **1C.** FB

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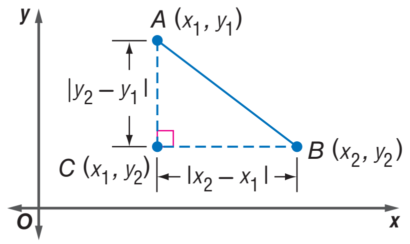
To find the distance between two points A and B in the coordinate plane, you can form a right triangle with $\overline{AB}$ as its hypotenuse and point C as its vertex as shown. Then use the Pythagorean Theorem to find AB.



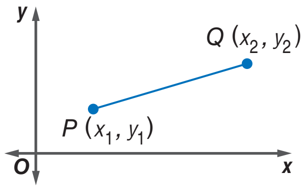
$(CB)^2 + (AC)^2 = (AB)^2$	Pythagorean Theorem
$(\lvert x_2 - x_1 \rvert)^2 + (\lvert y_2 - y_1 \rvert)^2 = (AB)^2$	$CB = \lvert x_2 - x_1 \rvert$, $AC = \lvert y_2 - y_1 \rvert$
$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (AB)^2$	The square of a number is always positive.
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = AB$	Take the positive square root of each side.

This gives us a Distance Formula for points in the coordinate plane. Because this formula involves taking the square root of a real number, distances can be irrational. Recall that an **irrational number** is a number that cannot be expressed as a terminating or repeating decimal.

KeyConcept Distance Formula (in Coordinate Plane)

If P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2), then

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



The order of the x- and y-coordinates in each set of parentheses is not important.

Example 2 Find Distance on a Coordinate Plane

Find the distance between $C(-4, -6)$ and $D(5, -1)$.

$CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{[5 - (-4)]^2 + [-1 - (-6)]^2}$	$(x_1, y_1) = (-4, -6)$ and $(x_2, y_2) = (5, -1)$
$= \sqrt{9^2 + 5^2}$ or $\sqrt{106}$	Subtract.

The distance between C and D is $\sqrt{106}$ units. Use a calculator to find that $\sqrt{106}$ units is approximately 10.3 units.

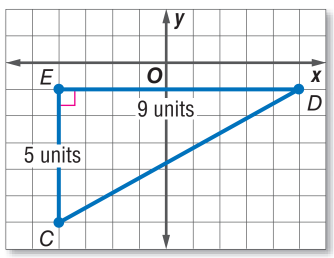
CHECK Graph the ordered pairs and check by using the Pythagorean Theorem.

$(CD)^2 \stackrel{?}{=} (EC)^2 + (ED)^2$

$(CD)^2 \stackrel{?}{=} 5^2 + 9^2$

$(CD)^2 \stackrel{?}{=} 106$

$CD = \sqrt{106}$ ✓



GuidedPractice

Find the distance between each pair of points.

2A. $E(-5, 6)$ and $F(8, -4)$

2B. $J(4, 3)$ and $K(-3, -7)$

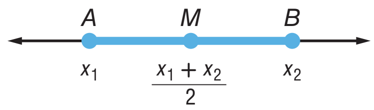
2 Midpoint of a Segment

The **midpoint** of a segment is the point halfway between the endpoints of the segment. If X is the midpoint of $\overline{AB}$, then $AX = XB$ and $\overline{AX} \cong \overline{XB}$. You can find the midpoint of a segment on a number line by finding the *mean*, or the average, of the coordinates of its endpoints.

KeyConcept Midpoint Formula (on Number Line)

If $\overline{AB}$ has endpoints at x_1 and x_2 on a number line, then the midpoint M of $\overline{AB}$ has coordinate

$$\frac{x_1 + x_2}{2}.$$

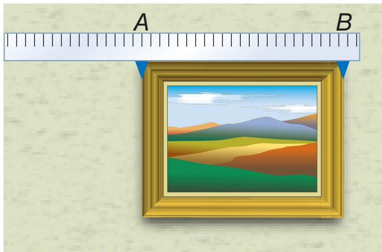


StudyTip

Alternative Method In Example 3, the coordinate of the midpoint could also have been located by first finding the length of AB, which is $37.5 - 15$ or 22.5 inches. Half of this measure is the distance from one endpoint to the point midway between A and B, $\frac{22.5}{2}$ or 11.25. Add this distance to point A's distance from the left wall. So the midpoint between A and B is $15 + 11.25$ or 26.25 inches from the left wall.

Real-World Example 3 Find Midpoint on a Number Line

DECORATING Jacinta hangs a picture 15 inches from the left side of a wall. How far from the edge of the wall should she mark the location for the nail the picture will hang on if the right edge is 37.5 inches from the wall's left side?



The coordinates of the endpoints of the top of the picture frame are 15 inches and 37.5 inches. Let M be the midpoint of $\overline{AB}$.

$M = \frac{x_1 + x_2}{2}$	Midpoint Formula
$= \frac{15 + 37.5}{2}$	$x_1 = 15$, $x_2 = 37.5$
$= \frac{52.5}{2}$ or 26.25	Simplify.

The midpoint is located at 26.25 or $26\frac{1}{4}$ inches from the left edge of the wall.

GuidedPractice

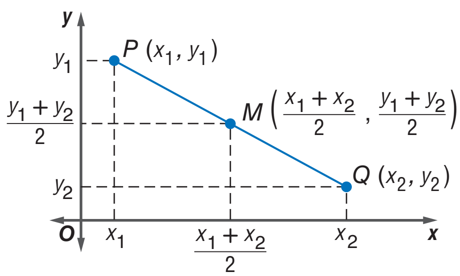
3. **TEMPERATURE** The temperature on a thermometer dropped from a reading of 25° to −8°. Find the midpoint of these temperatures.

You can find the midpoint of a segment on the coordinate plane by finding the average of the x-coordinates and of the y-coordinates of the endpoints.

KeyConcept Midpoint Formula (in Coordinate Plane)

If $\overline{PQ}$ has endpoints at $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the coordinate plane, then the midpoint M of $\overline{PQ}$ has coordinates

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



When finding the midpoint of a segment, the order of the coordinates of the endpoints is not important.

Example 4 Find Midpoint in Coordinate Plane

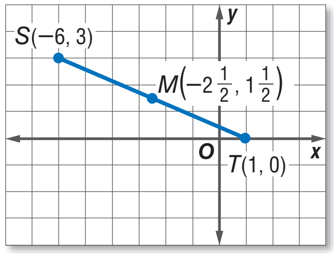
Find the coordinates of M, the midpoint of $\overline{ST}$, for $S(-6, 3)$ and $T(1, 0)$.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \quad \text{Midpoint Formula}$$

$$= \left(\frac{-6 + 1}{2}, \frac{3 + 0}{2}\right) \quad (x_1, y_1) = S(-6, 3), (x_2, y_2) = T(1, 0)$$

$$= \left(\frac{-5}{2}, \frac{3}{2}\right) \text{ or } M\left(-2\tfrac{1}{2}, 1\tfrac{1}{2}\right) \quad \text{Simplify.}$$

CHECK Graph S, T, and M. The distance from S to M does appear to be the same as the distance from M to T, so our answer is reasonable.



GuidedPractice

Find the coordinates of the midpoint of a segment with the given coordinates.

4A. $A(5, 12)$, $B(-4, 8)$

4B. $C(-8, -2)$, $D(5, 1)$

You can also find the coordinates of the endpoint of a segment if you know the coordinates of its other endpoint and its midpoint.

Example 5 Find the Coordinates of an Endpoint

Find the coordinates of J if $K(-1, 2)$ is the midpoint of $\overline{JL}$ and L has coordinates $(3, -5)$.

Step 1 Let J be (x_1, y_1) and L be (x_2, y_2) in the Midpoint Formula.

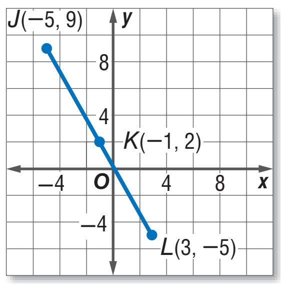
$$K\left(\frac{x_1 + 3}{2}, \frac{y_1 + (-5)}{2}\right) = K(-1, 2) \quad (x_2, y_2) = (3, -5)$$

Step 2 Write two equations to find the coordinates of J.

$\frac{x_1 + 3}{2} = -1$	Midpoint Formula	$\frac{y_1 + (-5)}{2} = 2$	Midpoint Formula
$x_1 + 3 = -2$	Multiply each side by 2.	$y_1 - 5 = 4$	Multiply each side by 2.
$x_1 = -5$	Subtract 3 from each side.	$y_1 = 9$	Add 5 to each side.

The coordinates of J are $(-5, 9)$.

CHECK Graph J, K, and L. The distance from J to K does appear to be the same as the distance from K to L, so our answer is reasonable.



GuidedPractice

Find the coordinates of the missing endpoint if P is the midpoint of $\overline{EG}$.

5A. $E(-8, 6)$, $P(-5, 10)$

5B. $P(-1, 3)$, $G(5, 6)$

StudyTip

Check for Reasonableness Always graph the given information and the calculated coordinates of the third point to check the reasonableness of your answer.

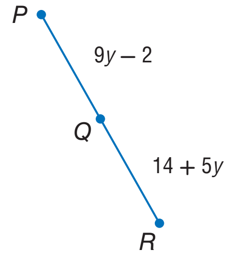
You can use algebra to find a missing measure or value in a figure that involves the midpoint of a segment.

Study Tip

CCSS Sense-Making and Perseverance The four-step problem solving plan is a tool for making sense of any problem. When making and executing your plan, continually ask yourself, "Does this make sense?" Monitor and evaluate your progress and change course if necessary.

Example 6 Use Algebra to Find Measures

ALGEBRA **Find the measure of $\overline{PQ}$ if Q is the midpoint of $\overline{PR}$.**



Understand You know that Q is the midpoint of $\overline{PR}$.
You are asked to find the measure of $\overline{PQ}$.

Plan Because Q is the midpoint, you know that $PQ = QR$. Use this equation to find a value for y.

Solve

$PQ = QR$	Definition of midpoint
$9y - 2 = 14 + 5y$	$PQ = 9y - 2$, $QR = 14 + 5y$
$4y - 2 = 14$	Subtract $5y$ from each side.
$4y = 16$	Add 2 to each side.
$y = 4$	Divide each side by 4.

Now substitute 4 for y in the expression for PQ.

$PQ = 9y - 2$	Original measure
$= 9(4) - 2$	$y = 4$
$= 36 - 2$ or 34	Simplify.

The measure of $\overline{PQ}$ is 34.

Check Since $PQ = QR$, when the expression for QR is evaluated for 4, it should also be 34.

$QR = 14 + 5y$	Original measure
$\stackrel{?}{=} 14 + 5(4)$	$y = 4$
$= 34$ ✓	Simplify.

Guided Practice

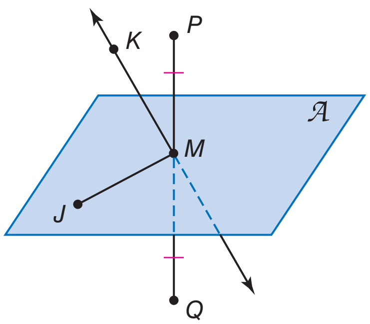
6A. Find the measure of $\overline{YZ}$ if Y is the midpoint of $\overline{XZ}$ and $XY = 2x - 3$ and $YZ = 27 - 4x$.

6B. Find the value of x if C is the midpoint of $\overline{AB}$, $AC = 4x + 5$, and $AB = 78$.

Study Tip

Segment Bisectors There can be an infinite number of bisectors and each must contain the midpoint of the segment.

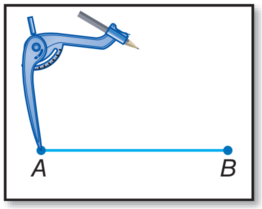
Any segment, line, or plane that intersects a segment at its midpoint is called a **segment bisector**. In the figure at the right, M is the midpoint of $\overline{PQ}$. Plane $\mathcal{A}$, $\overline{MJ}$, $\overleftrightarrow{KM}$, and point M are all bisectors of $\overline{PQ}$. We say that they *bisect* $\overline{PQ}$.



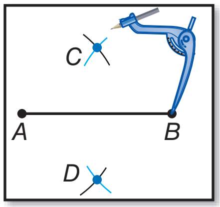
The construction on the following page shows how to construct a line that bisects a segment to find the midpoint of a given segment.

Construction Bisect a Segment

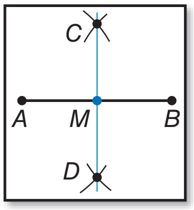
Step 1 Draw a segment and name it $\overline{AB}$. Place the compass at point *A*. Adjust the compass so that its width is greater than $\frac{1}{2}AB$. Draw arcs above and below $\overline{AB}$.



Step 2 Using the same compass setting, place the compass at point *B* and draw arcs above and below $\overline{AB}$ so that they intersect the two arcs previously drawn. Label the points of the intersection of the arcs as *C* and *D*.



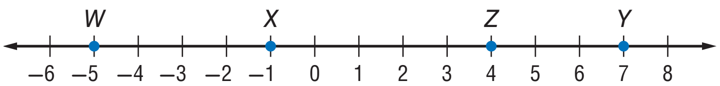
Step 3 Use a straightedge to draw $\overline{CD}$. Label the point where it intersects $\overline{AB}$ as *M*. Point *M* is the midpoint of $\overline{AB}$, and $\overline{CD}$ is a bisector of $\overline{AB}$.



Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

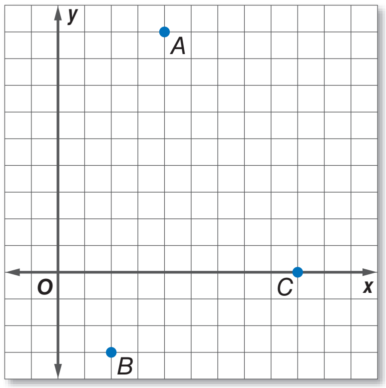
Example 1 **Use the number line to find each measure.**



1. *XY*

2. *WZ*

Example 2 **TIME CAPSULE** Graduating classes have buried time capsules on the campus of East Side High School for over twenty years. The points on the diagram show the position of three time capsules. Find the distance between each pair of time capsules.



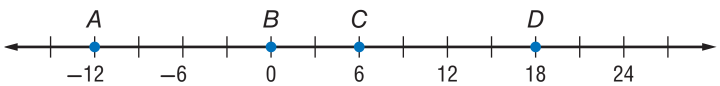
3. $A(4, 9)$, $B(2, -3)$

4. $A(4, 9)$, $C(9, 0)$

5. $B(2, -3)$, $C(9, 0)$

6. CCSS **REASONING** Which two time capsules are the closest to each other? Which are farthest apart?

Example 3 **Use the number line to find the coordinate of the midpoint of each segment.**



7. $\overline{AC}$

8. $\overline{BD}$

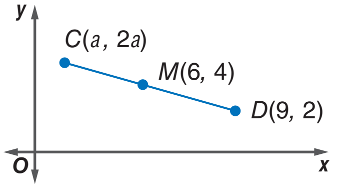
Example 4 **Find the coordinates of the midpoint of a segment with the given endpoints.**

9. $J(5, -3)$, $K(3, -8)$

10. $M(7, 1)$, $N(4, -1)$

Example 5 **11** Find the coordinates of G if $F(1, 3.5)$ is the midpoint of $\overline{GJ}$ and J has coordinates $(6, -2)$.

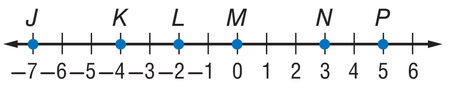
Example 6 **12.** **ALGEBRA** Point M is the midpoint of $\overline{CD}$. What is the value of a in the figure?



Practice and Problem Solving

Extra Practice is on page R1.

Example 1 **Use the number line to find each measure.**

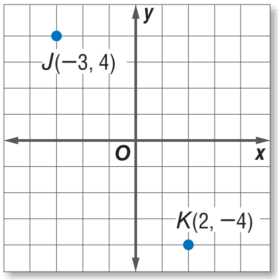


13. JL **14.** JK **15.** KP

16. NP **17.** JP **18.** LN

Example 2 **Find the distance between each pair of points.**

19.



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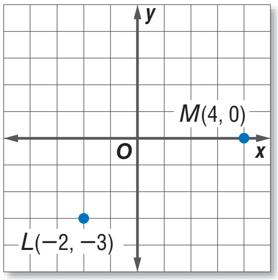

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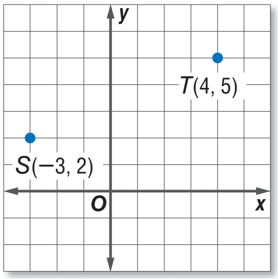

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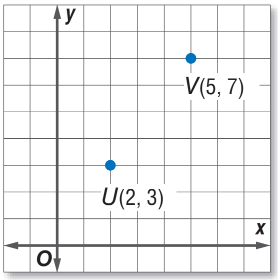

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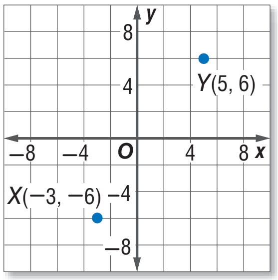

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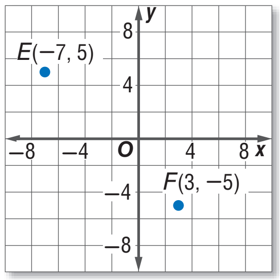
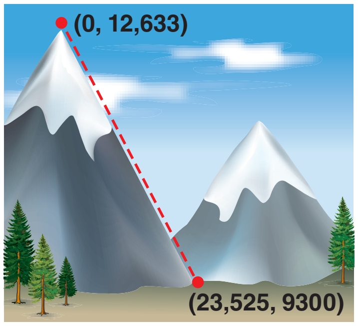


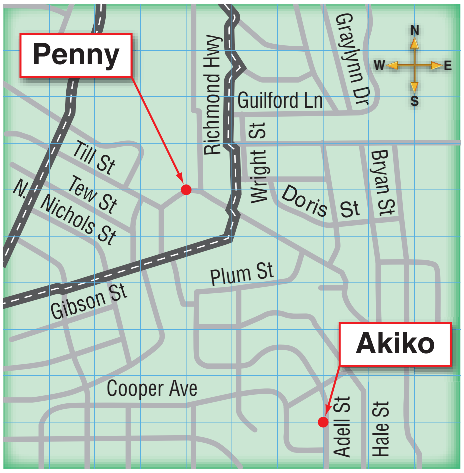
25. $X(1, 2)$, $Y(5, 9)$ **26.** $P(3, 4)$, $Q(7, 2)$ **27.** $M(-3, 8)$, $N(-5, 1)$

28. $Y(-4, 9)$, $Z(-5, 3)$ **29.** $A(2, 4)$, $B(5, 7)$ **30.** $C(5, 1)$, $D(3, 6)$

31. **CCSS REASONING** Vivian is planning to hike to the top of Humphreys Peak on her family vacation. The coordinates of the peak of the mountain and of the base of the trail are shown. If the trail can be approximated by a straight line, estimate the length of the trail. (*Hint:* 1 mi = 5280 ft)



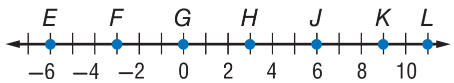
32. **CCSS MODELING** Penny and Akiko live in the locations shown on the map below.



a. If each square on the grid represents one block and the bottom left corner of the grid is the location of the origin, what is the straight-line distance from Penny's house to Akiko's?

b. If Penny moves three blocks to the north and Akiko moves 5 blocks to the west, how far apart will they be?

Example 3 **Use the number line to find the coordinate of the midpoint of each segment.**



33. $\overline{HK}$	**34.** $\overline{JL}$	**35.** $\overline{EF}$
36. $\overline{FG}$	**37.** $\overline{FK}$	**38.** $\overline{EL}$

Example 4 **Find the coordinates of the midpoint of a segment with the given endpoints.**

39. $C(22, 4)$, $B(15, 7)$

40. $W(12, 2)$, $X(7, 9)$

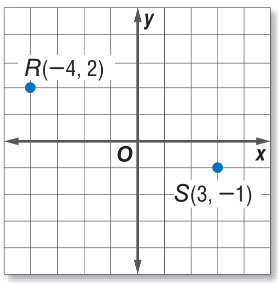
41. $D(-15, 4)$, $E(2, -10)$

42. $V(-2, 5)$, $Z(3, -17)$

43. $X(-2.4, -14)$, $Y(-6, -6.8)$

44. $J(-11.2, -3.4)$, $K(-5.6, -7.8)$

45.



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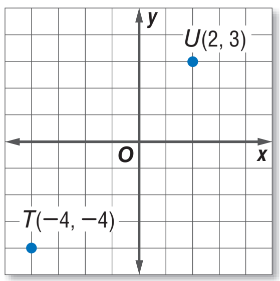


Example 5 **Find the coordinates of the missing endpoint if *B* is the midpoint of $\overline{AC}$.**

47. $C(-5, 4)$, $B(-2, 5)$	**48.** $A(1, 7)$, $B(-3, 1)$	**49.** $A(-4, 2)$, $B(6, -1)$
50. $C(-6, -2)$, $B(-3, -5)$	**51.** $A(4, -0.25)$, $B(-4, 6.5)$	**52.** $C\left(\frac{5}{3}, -6\right)$, $B\left(\frac{8}{3}, 4\right)$

Example 6 **ALGEBRA** **Suppose *M* is the midpoint of $\overline{FG}$. Use the given information to find the missing measure or value.**

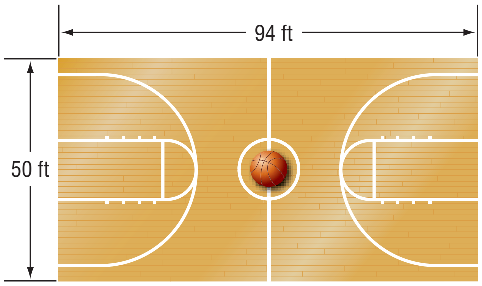
53. $FM = 3x - 4$, $MG = 5x - 26$, $FG = ?$

54. $FM = 5y + 13$, $MG = 5 - 3y$, $FG = ?$

55. $MG = 7x - 15$, $FG = 33$, $x = ?$

56. $FM = 8a + 1$, $FG = 42$, $a = ?$

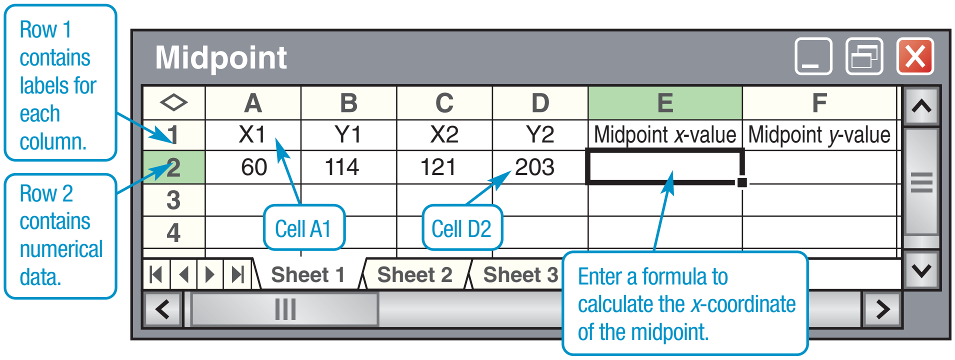
57 **BASKETBALL** The dimensions of a basketball court are shown below. Suppose a player throws the ball from a corner to a teammate standing at the center of the court.



a. If center court is located at the origin, find the ordered pair that represents the location of the player in the bottom right corner.

b. Find the distance that the ball travels.

CCSS TOOLS **Spreadsheets can be used to perform calculations quickly. The spreadsheet below can be used to calculate the distance between two points. Values are used in formulas by using a specific cell name. The value of x_1 is used in a formula using its cell name, A2.**



Write a formula for the indicated cell that could be used to calculate the indicated value using the coordinates (x_1, y_1) and (x_2, y_2) as the endpoint of a segment.

58. E2; the x-value of the midpoint of the segment

59. F2; the y-value of the midpoint of the segment

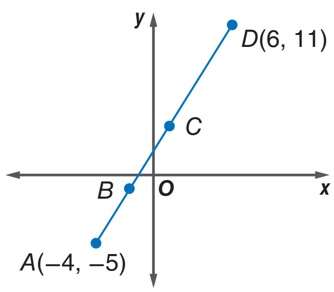
60. G2; the length of the segment

Name the point(s) that satisfy the given condition.

61. two points on the x-axis that are 10 units from (1, 8)

62. two points on the y-axis that are 25 units from (−24, 3)

63. **COORDINATE GEOMETRY** Find the coordinates of B if B is the midpoint of $\overline{AC}$ and C is the midpoint of $\overline{AD}$.

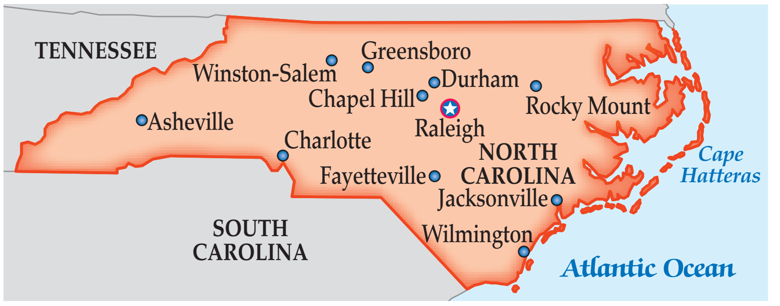


ALGEBRA **Determine the value(s) of n.**

64. $J(n, n + 2)$, $K(3n, n - 1)$, $JK = 5$

65. $P(3n, n - 7)$, $Q(4n, n + 5)$, $PQ = 13$

66. CCSS **PERSEVERANCE** Wilmington, North Carolina, is located at (34.3°, 77.9°), which represents north latitude and west longitude. Winston-Salem is in the northern part of the state at (36.1°, 80.2°).



a. Find the latitude and longitude of the midpoint of the segment between Wilmington and Winston-Salem.

b. Use an atlas or the Internet to find a city near the location of the midpoint.

c. If Winston-Salem is the midpoint of the segment with one endpoint at Wilmington, find the latitude and longitude of the other endpoint.

d. Use an atlas or the Internet to find a city near the location of the other endpoint.

67 **MULTIPLE REPRESENTATIONS** In this problem, you will explore the relationship between a midpoint of a segment and the midpoint between the endpoint and the midpoint.

a. Geometric Use a straightedge to draw three different line segments. Label the endpoints A and B.

b. Geometric On each line segment, find the midpoint of $\overline{AB}$ and label it C. Then find the midpoint of $\overline{AC}$ and label it D.

c. Tabular Measure and record AB, AC, and AD for each line segment. Organize your results into a table.

d. Algebraic If $AB = x$, write an expression for the measures AC and AD.

e. Verbal Make a conjecture about the relationship between AB and each segment if you were to continue to find the midpoints of a segment and a midpoint you previously found.

H.O.T. Problems Use Higher-Order Thinking Skills

68. **WRITING IN MATH** Explain how the Pythagorean Theorem and the Distance Formula are related.

69. **REASONING** Is the point one third of the way from (x_1, y_1) to (x_2, y_2) *sometimes, always,* or *never* the point $\left(\frac{x_1 + x_2}{3}, \frac{y_1 + y_2}{3}\right)$? Explain.

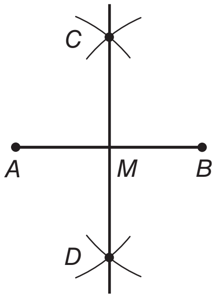
70. **CHALLENGE** Point P is located on the segment between point $A(1, 4)$ and point $D(7, 13)$. The distance from A to P is twice the distance from P to D. What are the coordinates of point P?

71. **OPEN ENDED** Draw a segment and name it $\overline{AB}$. Using only a compass and a straightedge, construct a segment $\overline{CD}$ such that $CD = 5\frac{1}{4}AB$. Explain and then justify your construction.

72. **WRITING IN MATH** Describe a method of finding the midpoint of a segment that has one endpoint at (0, 0). Give an example using your method, and explain why your method works.

Standardized Test Practice

73. Which of the following best describes the first step in bisecting $\overline{AB}$?



A From point A, draw equal arcs on $\overline{CD}$ using the same compass width.

B From point A, draw equal arcs above and below $\overline{AB}$ using a compass width of $\frac{1}{3}\overline{AB}$.

C From point A, draw equal arcs above and below $\overline{AB}$ using a compass width greater than $\frac{1}{2}\overline{AB}$.

D From point A, draw equal arcs above and below $\overline{AB}$ using a compass width less than $\frac{1}{2}\overline{AB}$.

74. ALGEBRA Beth paid $74.88 for 3 pairs of jeans. All 3 pairs of jeans were the same price. How much did each pair of jeans cost?

F $24.96 **H** $74.88

G $37.44 **J** $224.64

75. SAT/ACT If $5^{2x-3} = 1$, then $x =$

A 0.4 **D** 1.6

B 0.6 **E** 2

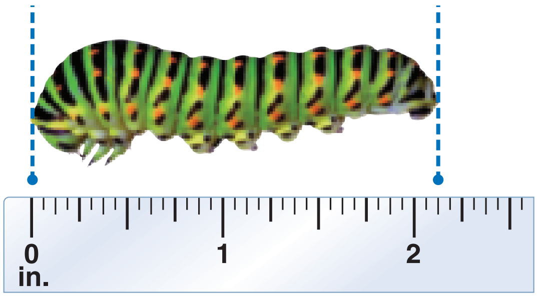
C 1.5

76. GRIDDED RESPONSE One endpoint of $\overline{AB}$ has coordinates $(-3, 5)$. If the coordinates of the midpoint of $\overline{AB}$ are $(2, -6)$, what is the approximate length of $\overline{AB}$?

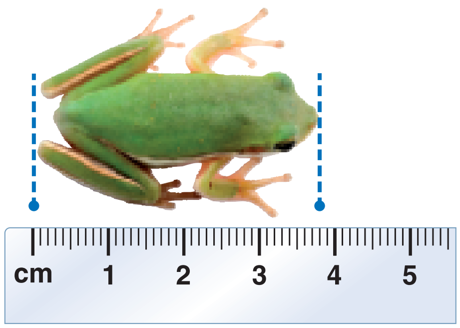
Spiral Review

Find the length of each object. (Lesson 1-2)

77.



78.



Draw and label a figure for each relationship. (Lesson 1-1)

79. $\overleftrightarrow{FG}$ lies in plane M and contains point H.

80. Lines r and s intersect at point W.

81. TRUCKS A sport-utility vehicle has a maximum load limit of 75 pounds for its roof. You want to place a 38-pound cargo carrier and 4 pieces of luggage on top of the roof. Write and solve an inequality to find the average allowable weight for each piece of luggage. (Lesson 0-6)

Skills Review

Solve each equation.

82. $8x - 15 = 5x$

83. $5y - 3 + y = 90$

84. $16a + 21 = 20a - 9$

85. $9k - 7 = 21 - 3k$

86. $11z - 13 = 3z + 17$

87. $15 + 6n = 4n + 23$