LESSON 10-2

Measuring Angles and Arcs

Then
- You measured angles and identified congruent angles.

Now
1. Identify central angles, major arcs, minor arcs, and semicircles, and find their measures.
2. Find arc lengths.

Why?
- The thirteen stars of the Betsy Ross flag are arranged equidistant from each other and from a fixed point. The distance between consecutive stars varies depending on the size of the flag, but the measure of the central angle formed by the center of the circle and any two consecutive stars is always the same.

NewVocabulary
- central angle
- arc
- minor arc
- major arc
- semicircle
- congruent arcs
- adjacent arcs
- arc length

Common Core State Standards

Content Standards

G.C.2 Identify and describe relationships among inscribed angles, radii, and chords.

G.C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

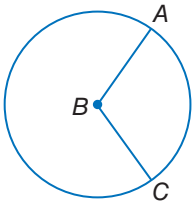
Mathematical Practices

6 Attend to precision.

4 Model with mathematics.

1 Angles and Arcs

A **central angle** of a circle is an angle with a vertex in the center of the circle. Its sides contain two radii of the circle. $\angle ABC$ is a central angle of $\odot B$.

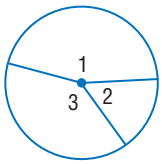


Recall from Lesson 1-4 that a *degree* is $\frac{1}{360}$ of the circular rotation about a point. This leads to the following relationship.

KeyConcept Sum of Central Angles

Words The sum of the measures of the central angles of a circle with no interior points in common is 360.

Example $m\angle 1 + m\angle 2 + m\angle 3 = 360$



Example 1 Find Measures of Central Angles

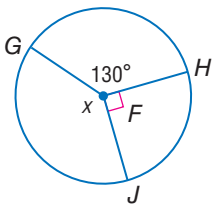
Find the value of x.

$m\angle GFH + m\angle HFJ + m\angle GFJ = 360$ Sum of Central Angles

$130 + 90 + m\angle GFJ = 360$ Substitution

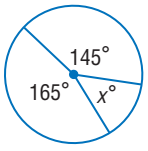
$220 + m\angle GFJ = 360$ Simplify.

$m\angle GFJ = 140$ Subtract 220 from each side.

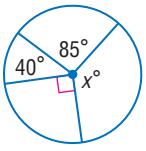


GuidedPractice

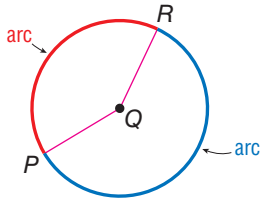
1A.



1B.



An **arc** is a portion of a circle defined by two endpoints. A central angle separates the circle into two arcs with measures related to the measure of the central angle.

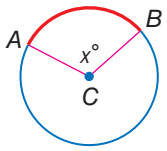
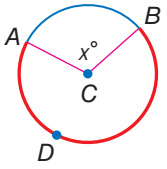
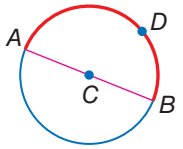


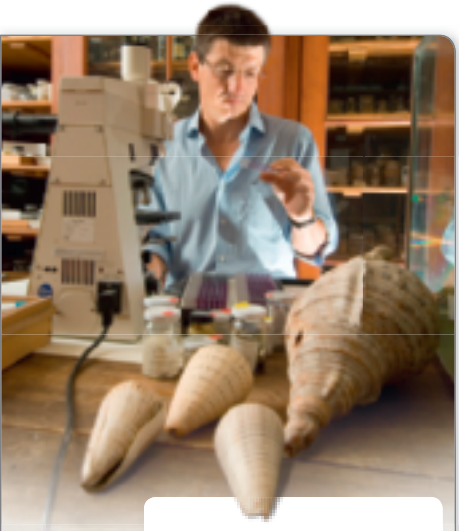
Photodisc/Getty Images

StudyTip

Naming Arcs Minor arcs are named by their endpoints. Major arcs and semicircles are named by their endpoints and another point on the arc that lies between these endpoints.

KeyConcept Arcs and Arc Measure

Arc	Measure	
A **minor arc** is the shortest arc connecting two endpoints on a circle.	The measure of a minor arc is less than 180 and equal to the measure of its related central angle. $m\overset{\frown}{AB} = m\angle ACB = x$	
A **major arc** is the longest arc connecting two endpoints on a circle.	The measure of a major arc is greater than 180, and equal to 360 minus the measure of the minor arc with the same endpoints. $m\overset{\frown}{ADB} = 360 - m\overset{\frown}{AB} = 360 - x$	
A **semicircle** is an arc with endpoints that lie on a diameter.	The measure of a semicircle is 180. $m\overset{\frown}{ADB} = 180$	



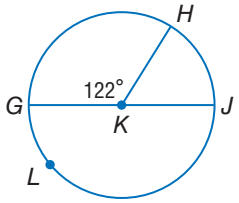
Real-WorldCareer

Historical Researcher

Research in museums includes authentication, verification, and description of artifacts. Employment as a historical researcher requires a minimum of a bachelor's degree in history. Refer to Exercises 42–43.

Example 2 Classify Arcs and Find Arc Measures

$\overline{GJ}$ **is a diameter of** $\odot K$**. Identify each arc as a** *major arc, minor arc,* **or** *semicircle.* **Then find its measure.**



a. $m\overset{\frown}{GH}$

$\overset{\frown}{GH}$ is a minor arc, so $m\overset{\frown}{GH} = m\angle GKH$ or 122.

b. $m\overset{\frown}{GLH}$

$\overset{\frown}{GLH}$ is a major arc that shares the same endpoints as minor arc $\overset{\frown}{GH}$.

$m\overset{\frown}{GHL} = 360 - m\overset{\frown}{GH}$

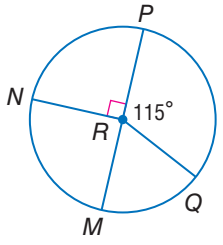
$= 360 - 122$ or 238

c. $m\overset{\frown}{GLJ}$

$\overset{\frown}{GLJ}$ is a semicircle, so $m\overset{\frown}{GLJ} = 180$.

GuidedPractice

$\overline{PM}$ **is a diameter of** $\odot R$**. Identify each arc as a** *major arc, minor arc,* **or** *semicircle.* **Then find its measure.**



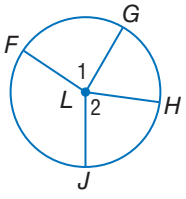
2A. $\overset{\frown}{MQ}$ **2B.** $\overset{\frown}{MNP}$ **2C.** $\overset{\frown}{MNQ}$

Congruent arcs are arcs in the same or congruent circles that have the same measure.

Theorem 10.1

Words In the same circle or in congruent circles, two minor arcs are congruent if and only if their central angles are congruent.

Example If $\angle 1 \cong \angle 2$, then $\overset{\frown}{FG} \cong \overset{\frown}{HJ}$.
If $\overset{\frown}{FG} \cong \overset{\frown}{HJ}$, then $\angle 1 \cong \angle 2$.

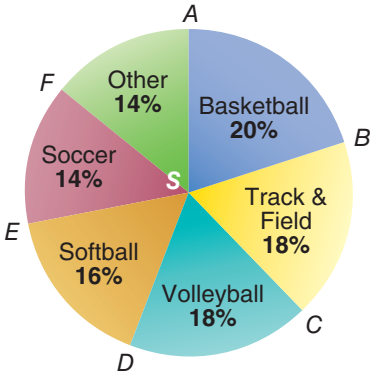


You will prove Theorem 10.1 in Exercise 52.

Real-World Example 3 Find Arc Measures in Circle Graphs

SPORTS Refer to the circle graph. Find each measure.

Female Participation in Sports



a. $m\overset{\frown}{CD}$

$\overset{\frown}{CD}$ is a minor arc. $m\overset{\frown}{CD} = m\angle CSD$

$\angle CSD$ represents 18% of the whole, or 18% of the circle.

$m\angle CSD = 0.18(360)$ Find 18% of 360.

$= 64.8$ Simplify.

b. $m\overset{\frown}{BC}$

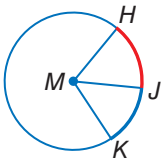
The percents for volleyball and track and field are equal, so the central angles are congruent and the corresponding arcs are congruent.

$m\overset{\frown}{BC} = m\overset{\frown}{CD} = 64.8$

GuidedPractice

3A. $m\overset{\frown}{EF}$ **3B.** $m\overset{\frown}{FA}$

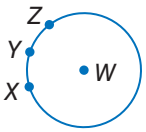
Adjacent arcs are arcs in a circle that have exactly one point in common. In $\odot M$, $\overset{\frown}{HJ}$ and $\overset{\frown}{JK}$ are adjacent arcs. As with adjacent angles, you can add the measures of adjacent arcs.



Postulate 10.1 Arc Addition Postulate

Words The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs.

Example $m\overset{\frown}{XYZ} = m\overset{\frown}{XY} + m\overset{\frown}{YZ}$

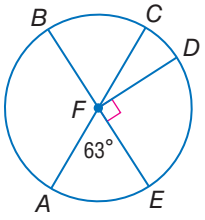


Math History Link

Euclid (c. 325–265 B.C.) The 13 books of Euclid's *Elements* are influential works of science. In them, geometry and other branches of mathematics are logically developed. Book 3 of *Elements* is devoted to circles, arcs, and angles.

Example 4 Use Arc Addition to Find Measures of Arcs

Find each measure in $\odot F$.



a. $m\overset{\frown}{AED}$

$m\overset{\frown}{AED} = m\overset{\frown}{AE} + m\overset{\frown}{ED}$ Arc Addition Postulate

$= m\angle AFE + m\angle EFD$ $m\overset{\frown}{AE} = m\angle AFE$, $m\overset{\frown}{ED} = m\angle EFD$

$= 63 + 90$ or 153 Substitution

b. $m\overset{\frown}{ADB}$

$m\overset{\frown}{ADB} = m\overset{\frown}{AE} + m\overset{\frown}{EDB}$ Arc Addition Postulate

$= 63 + 180$ or 243 $\overset{\frown}{EDB}$ is a semicircle, so $m\overset{\frown}{EDB} = 180$.

GuidedPractice

4A. $m\overset{\frown}{CE}$ **4B.** $m\overset{\frown}{ABD}$

Hulton Archive/Getty Images

WatchOut!

Arc Length The length of an arc is given in linear units, such as centimeters. The measure of an arc is given in degrees.

2 Arc Length

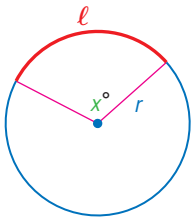
Arc length is the distance between the endpoints along an arc measured in linear units. Since an arc is a portion of a circle, its length is a fraction of the circumference.

KeyConcept Arc Length

Words The ratio of the length of an arc ℓ to the circumference of the circle is equal to the ratio of the degree measure of the arc to 360.

Proportion $\frac{\ell}{2\pi r} = \frac{x}{360}$ or

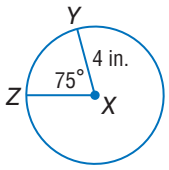
Equation $\ell = \frac{x}{360} \cdot 2\pi r$



Example 5 Find Arc Length

Find the length of $\overset{\frown}{ZY}$. Round to the nearest hundredth.

a.

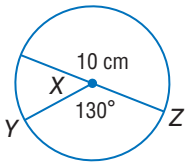


$\ell = \frac{x}{360} \cdot 2\pi r$ Arc Length Equation

$= \frac{75}{360} \cdot 2\pi(4)$ Substitution

≈ 5.24 in. Use a calculator.

b.

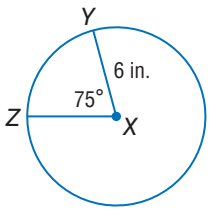


$\ell = \frac{x}{360} \cdot 2\pi r$ Arc Length Equation

$= \frac{130}{360} \cdot 2\pi(5)$ Substitution

≈ 11.34 cm Use a calculator.

c.



$\ell = \frac{x}{360} \cdot 2\pi r$ Arc Length Equation

$= \frac{75}{360} \cdot 2\pi(6)$ Substitution

≈ 7.85 in. Use a calculator.

StudyTip

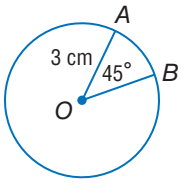
Alternate Method The arc lengths in Examples 5a, 5b, and 5c could also have been calculated using the arc length proportion $\frac{\ell}{2\pi r} = \frac{x}{360}$.

Notice that $\overset{\frown}{ZY}$ has the same measure, 75, in both Examples 5a and 5c. The arc lengths, however, are different. This is because they are in circles that have different radii.

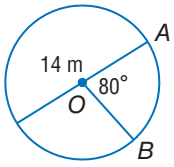
GuidedPractice

Find the length of $\overset{\frown}{AB}$. Round to the nearest hundredth.

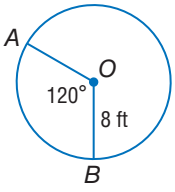
5A.



5B.



5C.

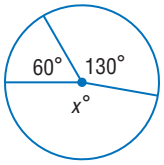


Check Your Understanding

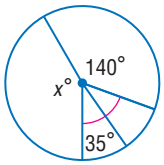
= Step-by-Step Solutions begin on page R14.

Example 1 Find the value of x.

1.



2.

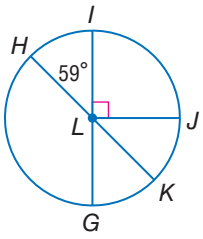


Example 2 CCSS **PRECISION** $\overline{HK}$ and $\overline{IG}$ are diameters of $\odot L$. Identify each arc as a *major arc, minor arc,* or *semicircle*. Then find its measure.

3. $m\widehat{IHJ}$

4. $m\widehat{HI}$

5. $m\widehat{HGK}$



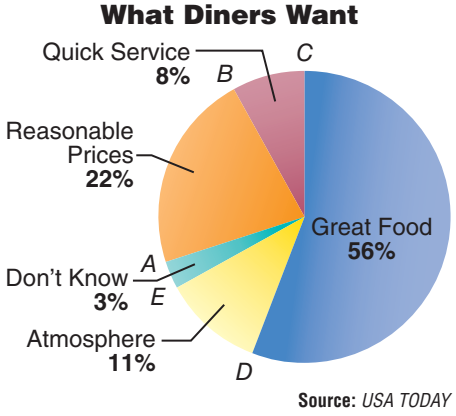
Example 3

6. **RESTAURANTS** The graph shows the results of a survey taken by diners relating what is most important about the restaurants where they eat.

   a. Find $m\widehat{AB}$.

   b. Find $m\widehat{BC}$.

   c. Describe the type of arc that the category Great Food represents.

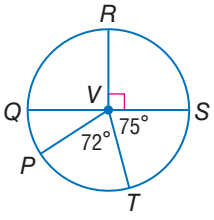


Example 4 $\overline{QS}$ is a diameter of $\odot V$. Find each measure.

7. $m\widehat{STP}$

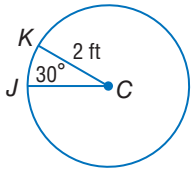
8. $m\widehat{QRT}$

9. $m\widehat{PQR}$

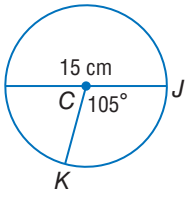


Example 5 Find the length of $\widehat{JK}$. Round to the nearest hundredth.

10.



11.

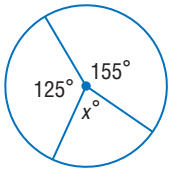


Practice and Problem Solving

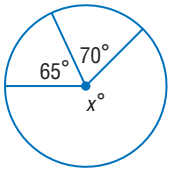
Extra Practice is on page R10.

Example 1 Find the value of x.

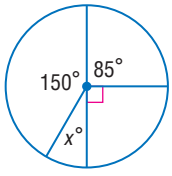
12.



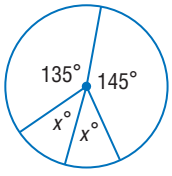
13.



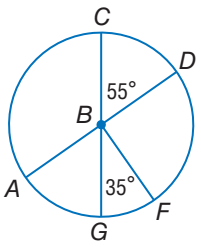
14.



15.



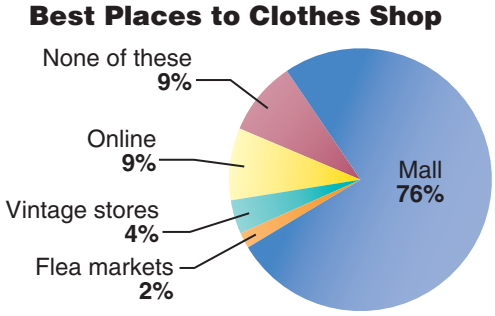
Example 2 $\overline{AD}$ and $\overline{CG}$ are diameters of $\odot B$. Identify each arc as a *major arc, minor arc,* or *semicircle*. Then find its measure.



16. $m\widehat{CD}$
17. $m\widehat{AC}$
18. $m\widehat{CFG}$
19. $m\widehat{CGD}$
20. $m\widehat{GCF}$
21. $m\widehat{ACD}$
22. $m\widehat{AG}$
23. $m\widehat{ACF}$

Example 3

24. **SHOPPING** The graph shows the results of a survey in which teens were asked where the best place was to shop for clothes.



   a. What would be the arc measures associated with the mall and vintage stores categories?

   b. Describe the kinds of arcs associated with the category "Mall" and the category "None of these."

   c. Are there any congruent arcs in this graph? Explain.

25. CCSS **MODELING** The table shows the results of a survey in which Americans were asked how long food could be on the floor and still be safe to eat.

Dropped Food	
Do you eat food dropped on the floor?	
Not safe to eat	78%
Three-second rule*	10%
Five-second rule*	8%
Ten-second rule*	4%

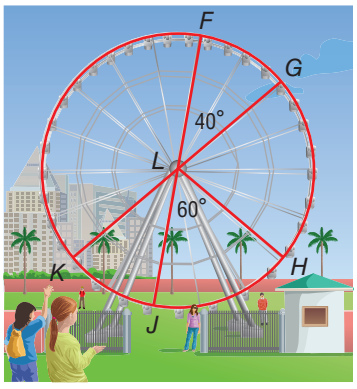
Source: American Diabetic Association
* The length of time the food is on the floor.

   a. If you were to construct a circle graph of this information, what would be the arc measures associated with the first two categories?

   b. Describe the kind of arcs associated with the first category and the last category.

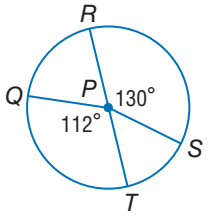
   c. Are there any congruent arcs in this graph? Explain.

Examples 2, 4 **ENTERTAINMENT** **Use the Ferris wheel shown to find each measure.**



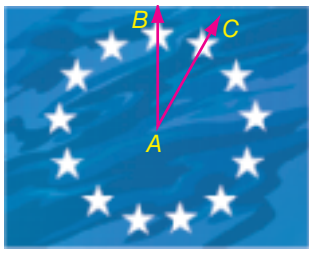
26. $m\widehat{FG}$
27. $m\widehat{JH}$
28. $m\widehat{JKF}$
29. $m\widehat{JFH}$
30. $m\widehat{GHF}$
31. $m\widehat{GHK}$
32. $m\widehat{HK}$
33. $m\widehat{JKG}$
34. $m\widehat{KFH}$
35. $m\widehat{HGF}$

Example 5 **Use $\odot P$ to find the length of each arc. Round to the nearest hundredth.**



36. $\widehat{RS}$, if the radius is 2 inches
37. $\widehat{QT}$, if the diameter is 9 centimeters
38. $\widehat{QR}$, if $PS = 4$ millimeters
39. $\widehat{RS}$, if $RT = 15$ inches
40. $\widehat{QRS}$, if $RT = 11$ feet
41. $\widehat{RTS}$, if $PQ = 3$ meters

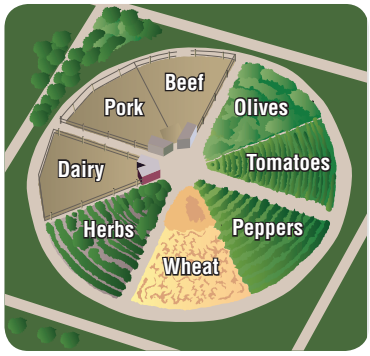
HISTORY The figure shows the stars in the Betsy Ross flag referenced at the beginning of the lesson.



42. What is the measure of central angle A? Explain how you determined your answer.

43. If the diameter of the circle were doubled, what would be the effect on the arc length from the center of one star B to the next star C?

44. FARMS The *Pizza Farm* in Madera, California, is a circle divided into eight equal slices, as shown at the right. Each "slice" is used for growing or grazing pizza ingredients.

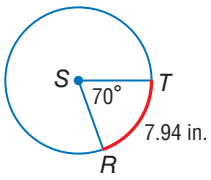


a. What is the total arc measure of the slices containing olives, tomatoes, and peppers?

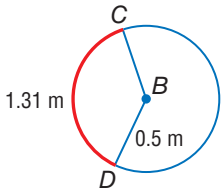
b. The circle is 125 feet in diameter. What is the arc length of one slice? Round to the nearest hundredth.

CCSS REASONING Find each measure. Round each linear measure to the nearest hundredth and each arc measure to the nearest degree.

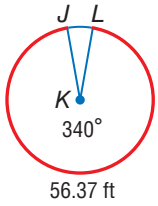
45. circumference of $\odot S$



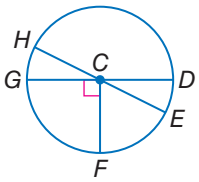
46. $m\overset{\frown}{CD}$



47. radius of $\odot K$



ALGEBRA In $\odot C$, $m\angle HCG = 2x$ and $m\angle HCD = 6x + 28$. Find each measure.

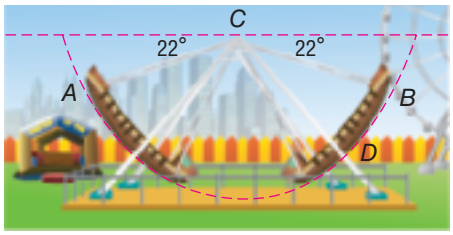


48. $m\overset{\frown}{EF}$

49. $m\overset{\frown}{HD}$

50. $m\overset{\frown}{HGF}$

51 RIDES A pirate ship ride follows a semicircular path, as shown in the diagram.



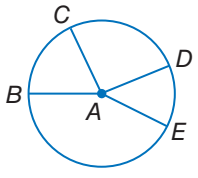
a. What is $m\overset{\frown}{AB}$?

b. If $CD = 62$ feet, what is the length of $\overset{\frown}{AB}$? Round to the nearest hundredth.

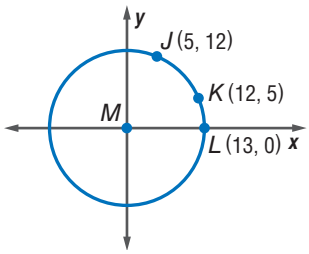
52. PROOF Write a two-column proof of Theorem 10.1.

Given: $\angle BAC \cong \angle DAE$

Prove: $\overset{\frown}{BC} \cong \overset{\frown}{DE}$



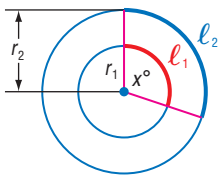
53 **COORDINATE GEOMETRY** In the graph, point M is located at the origin. Find each measure in $\odot M$. Round each linear measure to the nearest hundredth and each arc measure to the nearest tenth degree.



a. $m\widehat{JL}$ **b.** $m\widehat{KL}$ **c.** $m\widehat{JK}$

d. length of $\widehat{JL}$ **e.** length of $\widehat{JK}$

54. **ARC LENGTH AND RADIAN MEASURE** In this problem, you will use concentric circles to show that the length of the arc intercepted by a central angle of a circle is dependent on the circle's radius.



a. Compare the measures of arc ℓ_1 and arc ℓ_2. Then compare the lengths of arc ℓ_1 and arc ℓ_2. What do these two comparisons suggest?

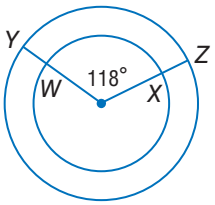
b. Use similarity transformations (dilations) to explain why the length of an arc ℓ intercepted by a central angle of a circle is proportional to the circle's radius r. That is, explain why we can say that for this diagram, $\frac{\ell_1}{r_1} = \frac{\ell_2}{r_2}$.

c. Write expressions for the lengths of arcs ℓ_1 and ℓ_2. Use these expressions to identify the constant of proportionality k in $\ell = kr$.

d. The expression that you wrote for k in part **c** gives the *radian measure* of an angle. Use it to find the radian measure of an angle measuring 90°.

H.O.T. Problems Use Higher-Order Thinking Skills

55. **ERROR ANALYSIS** Brody says that $\widehat{WX}$ and $\widehat{YZ}$ are congruent since their central angles have the same measure. Selena says they are not congruent. Is either of them correct? Explain your reasoning.



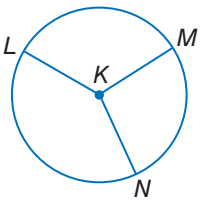
CCSS **ARGUMENTS** **Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your reasoning.**

56. The measure of a minor arc is less than 180.

57. If a central angle is obtuse, its corresponding arc is a major arc.

58. The sum of the measures of adjacent arcs of a circle depends on the measure of the radius.

59. **CHALLENGE** The measures of $\widehat{LM}$, $\widehat{MN}$, and $\widehat{NL}$ are in the ratio 5:3:4. Find the measure of each arc.



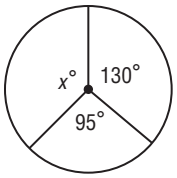
60. **OPEN ENDED** Draw a circle and locate three points on the circle. Estimate the measures of the three nonoverlapping arcs that are formed. Then use a protractor to find the measure of each arc. Label your circle with the arc measures.

61. **CHALLENGE** The time shown on an analog clock is 8:10. What is the measure of the angle formed by the hands of the clock?

62. **WRITING IN MATH** Describe the three different types of arcs in a circle and the method for finding the measure of each one.

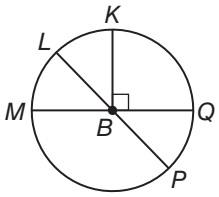
Standardized Test Practice

63. What is the value of x?



A 120
B 135
C 145
D 160

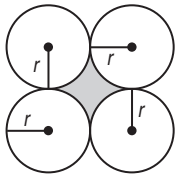
64. GRIDDED RESPONSE In $\odot B$, $m\angle LBM = 3x$ and $m\angle LBQ = 4x + 61$. What is the measure of $\angle PBQ$?



65. ALGEBRA A rectangle's width is represented by x and its length by y. Which expression best represents the area of the rectangle if the length and width are tripled?

F $3xy$
G $3(xy)^2$
H $9xy$
J $(xy)^3$

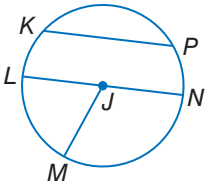
66. SAT/ACT What is the area of the shaded region if $r = 4$?



A $64 - 16\pi$
B $16 - 16\pi$
C $16 - 8\pi$
D $64 - 8\pi$
E $64\pi - 16$

Spiral Review

Refer to $\odot J$. (Lesson 10-1)



67. Name the center of the circle.

68. Identify a chord that is also a diameter.

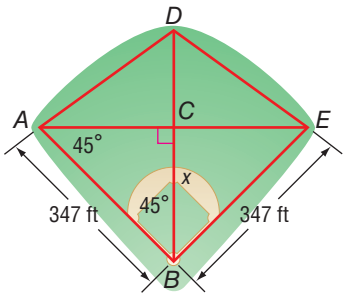
69. If $LN = 12.4$, what is JM?

Graph the image of each polygon with the given vertices after a dilation centered at the origin with the given scale factor. (Lesson 9-6)

70. $X(-1, 2)$, $Y(2, 1)$, $Z(-1, -2)$; $r = 3$

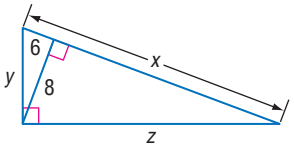
71. $A(-4, 4)$, $B(4, 4)$, $C(4, -4)$, $D(-4, -4)$; $r = 0.25$

72. BASEBALL The diagram shows some dimensions of Comiskey Park in Chicago, Illinois. $\overline{BD}$ is a segment from home plate to dead center field, and $\overline{AE}$ is a segment from the left field foul pole to the right field foul pole. If the center fielder is standing at C, how far is he from home plate? (Lesson 8-3)

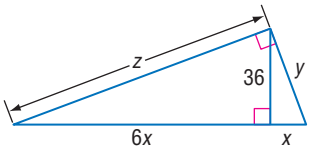


Find x, y, and z. (Lesson 8-1)

73.



74.



Skills Review

Find x.

75. $24^2 + x^2 = 26^2$

76. $x^2 + 5^2 = 13^2$

77. $30^2 + 35^2 = x^2$