LESSON

10-6 Secants, Tangents, and Angle Measures

Then

- You found measures of segments formed by tangents to a circle.

Now

1. Find measures of angles formed by lines intersecting on or inside a circle.
2. Find measures of angles formed by lines intersecting outside the circle.

Why?

- An average person's field of vision is about 180°. Most cameras have a much narrower viewing angle of between 20° and 50°. This viewing angle determines how much of a curved object a camera can capture on film.

NewVocabulary
secant

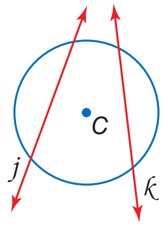
Common Core State Standards

Content Standards
Reinforcement of G.C.4 Construct a tangent line from a point outside a given circle to the circle.

Mathematical Practices
3 Construct viable arguments and critique the reasoning of others.
1 Make sense of problems and persevere in solving them.

1 Intersections On or Inside a Circle

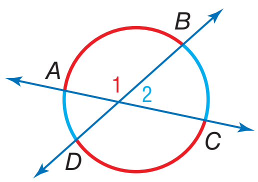
A **secant** is a line that intersects a circle in exactly two points. Lines j and k are secants of $\odot C$.



When two secants intersect inside a circle, the angles formed are related to the arcs they intercept.

Theorem 10.12

Words If two secants or chords intersect in the interior of a circle, then the measure of an angle formed is one half the *sum* of the measure of the arcs intercepted by the angle and its vertical angle.



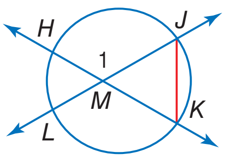
Example $m\angle 1 = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$ and $m\angle 2 = \frac{1}{2}(m\widehat{DA} + m\widehat{BC})$

Proof

Given: $\overleftrightarrow{HK}$ and $\overleftrightarrow{JL}$ intersect at M.

Prove: $m\angle 1 = \frac{1}{2}(m\widehat{JH} + m\widehat{LK})$

Proof:

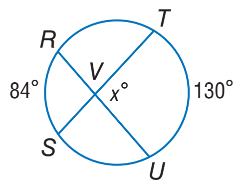


Statements	Reasons
1. $\overleftrightarrow{HK}$ and $\overleftrightarrow{JL}$ intersect at M.	1. Given
2. $m\angle 1 = m\angle MJK + m\angle MKJ$	2. Exterior Angle Theorem
3. $m\angle MJK = \frac{1}{2}m\angle\widehat{LK}$, $m\angle MKJ = \frac{1}{2}m\angle\widehat{JH}$	3. The measure of an inscribed ∠ equals half the measure of the intercepted arc.
4. $m\angle 1 = \frac{1}{2}m\angle\widehat{LK} + \frac{1}{2}m\angle\widehat{JH}$	4. Substitution
5. $m\angle 1 = \frac{1}{2}(m\widehat{JH} + m\widehat{LK})$	5. Distributive Property

Example 1 Use Intersecting Chords or Secants

Find x.

a.

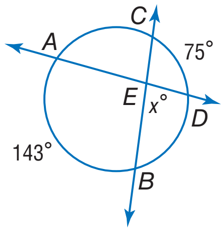


$m\angle TVU = \frac{1}{2}(m\widehat{RS} + m\widehat{TU})$ Theorem 10.12

$x = \frac{1}{2}(84 + 130)$ Substitution

$= \frac{1}{2}(214)$ or 107 Simplify.

b.



Step 1 Find $m\angle AEB$.

$m\angle AEB = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$ Theorem 10.12

$= \frac{1}{2}(143 + 75)$ Substitution

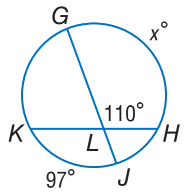
$= \frac{1}{2}(218)$ or 109 Simplify.

Step 2 Find x, the measure of $\angle DEB$.

$\angle AEB$ and $\angle DEB$ are supplementary angles.

So, $x = 180 - 109$ or 71.

c.



$m\angle GLH = \frac{1}{2}(m\widehat{GH} + m\widehat{KJ})$ Theorem 10.12

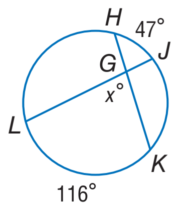
$110 = \frac{1}{2}(x + 97)$ Substitution

$220 = (x + 97)$ Multiply each side by 2.

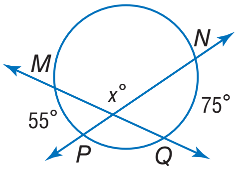
$123 = x$ Subtract 97 from each side.

Guided Practice

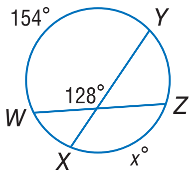
1A.



1B.



1C.



Study Tip

Alternative Method In Example 1b, $m\angle DEB$ can also be found by first finding the sum of the measures of $\widehat{AC}$ and $\widehat{BD}$.

$m\widehat{AC} + m\widehat{BD}$

$= 360 - (m\widehat{AC} + m\widehat{CD})$

$= 360 - (143 + 75)$

$= 142$

$m\angle DEB$

$= \frac{1}{2}(m\widehat{AC} + m\widehat{BD})$

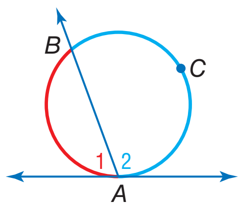
$= \frac{1}{2}(142)$ or 71

Recall that Theorem 10.6 states that the measure of an inscribed angle is half the measure of its intercepted arc. If one of the sides of this angle is tangent to the circle, this relationship still holds true.

Theorem 10.13

Words If a secant and a tangent intersect at the point of tangency, then the measure of each angle formed is one half the measure of its intercepted arc.

Example $m\angle 1 = \frac{1}{2}m\widehat{AB}$ and $m\angle 2 = \frac{1}{2}m\widehat{ACB}$



You will prove Theorem 10.13 in Exercise 33.

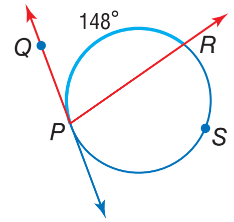
Example 2 Use Intersecting Secants and Tangents

Find each measure.

a. $m\angle QPR$

$m\angle QPR = \frac{1}{2}m\overset{\frown}{PR}$ Theorem 10.13

$= \frac{1}{2}(148)$ or 74   Substitute and simplify.



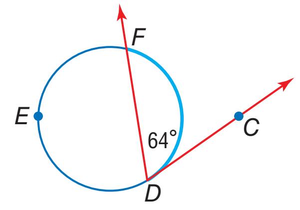
b. $m\overset{\frown}{DEF}$

$m\angle CDF = \frac{1}{2}m\overset{\frown}{FD}$ Theorem 10.13

$64 = \frac{1}{2}m\overset{\frown}{FD}$ Substitution

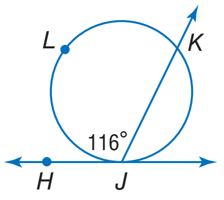
$128 = m\overset{\frown}{FD}$ Multiply each side by 2.

$m\overset{\frown}{DEF} = 360 - m\overset{\frown}{FD} = 360 - 128$ or 232

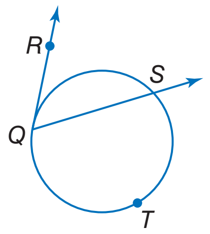


GuidedPractice

2A. Find $m\overset{\frown}{JLK}$.



2B. Find $m\angle RQS$ if $m\overset{\frown}{QTS} = 238$.



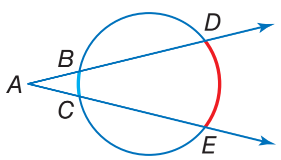
2 Intersections Outside a Circle

Secants and tangents can also meet outside a circle. The measure of the angle formed also involves half of the measures of the arcs they intercept.

Theorem 10.14

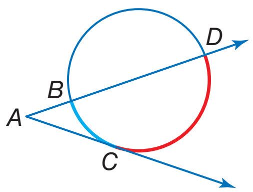
Words If two secants, a secant and a tangent, or two tangents intersect in the exterior of a circle, then the measure of the angle formed is one half the *difference* of the measures of the intercepted arcs.

Examples



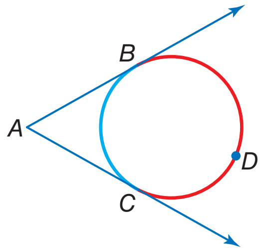
Two Secants

$m\angle A = \frac{1}{2}(m\overset{\frown}{DE} - m\overset{\frown}{BC})$



Secant-Tangent

$m\angle A = \frac{1}{2}(m\overset{\frown}{DC} - m\overset{\frown}{BC})$



Two Tangents

$m\angle A = \frac{1}{2}(m\overset{\frown}{BDC} - m\overset{\frown}{BC})$

You will prove Theorem 10.14 in Exercises 30–32.

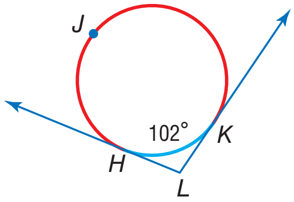
StudyTip

Absolute Value The measure of each $\angle A$ can also be expressed as half the absolute value of the difference of the arc measure. In this way, the order of the arc measures does not affect the outcome of the calculation.

Example 3 Use Tangents and Secants that Intersect Outside a Circle

Find each measure.

a. $m\angle L$

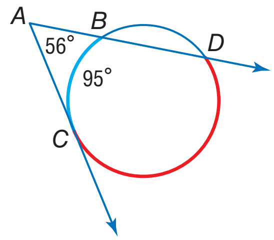


$m\angle L = \frac{1}{2}(m\widehat{HJK} - m\widehat{HK})$ Theorem 10.14

$= \frac{1}{2}(360 - 102) - 102$ Substitution

$= \frac{1}{2}(258 - 102)$ or 78 Simplify.

b. $m\widehat{CD}$



$m\angle A = \frac{1}{2}(m\widehat{CD} - m\widehat{BC})$ Theorem 10.14

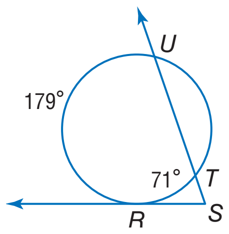
$56 = \frac{1}{2}(m\widehat{CD} - 95)$ Substitution

$112 = m\widehat{CD} - 95$ Multiply each side by 2.

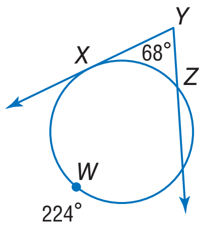
$207 = m\widehat{CD}$ Add 95 to each side.

GuidedPractice

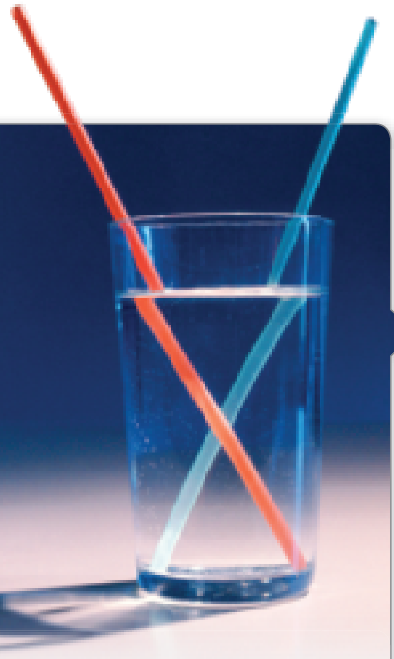
3A. $m\angle S$



3B. $m\widehat{XZ}$



You can apply the properties of intersecting secants to solve real-world problems.



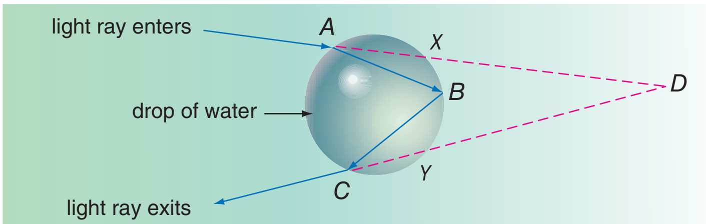
Real-WorldLink

There is a difference in the *index of refraction* between the two mediums such as air and glass. The index of refraction N is given by the equation $N = \frac{c}{V}$, where c is the speed of light and V is the velocity of light in that material.

Source: Microscopy Resource Center

Real-World Example 4 Apply Properties of Intersecting Secants

SCIENCE The diagram shows the path of a light ray as it hits a drop of water. The ray is bent, or *refracted*, at points A, B, and C. If $m\widehat{AC} = 128$ and $m\widehat{XBY} = 84$, what is $m\angle D$?



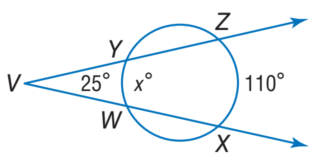
$m\angle D = \frac{1}{2}(m\widehat{AC} - m\widehat{XBY})$ Theorem 10.14

$= \frac{1}{2}(128 - 84)$ Substitution

$= \frac{1}{2}(44)$ or 22 Simplify.

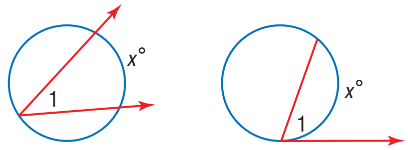
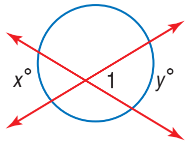
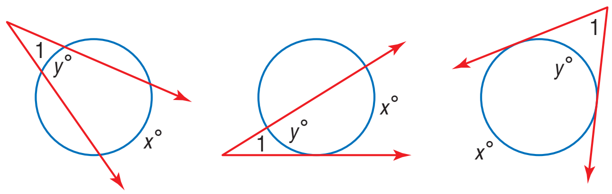
GuidedPractice

4. Find the value of x.



Ruddy Gold/age fotostock

KeyConcept Circle and Angle Relationships

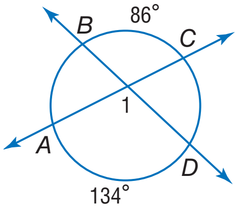
Vertex of Angle	Model(s)	Angle Measure
on the circle		one half the measure of the intercepted arc $m\angle 1 = \frac{1}{2}x$
inside the circle		one half the measure of the sum of the intercepted arc $m\angle 1 = \frac{1}{2}(x + y)$
outside the circle		one half the measure of the difference of the intercepted arcs $m\angle 1 = \frac{1}{2}(x - y)$

Check Your Understanding

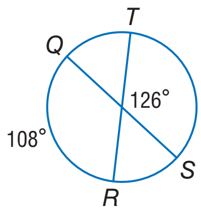
= Step-by-Step Solutions begin on page R14.

Examples 1–2 Find each measure. Assume that segments that appear to be tangent are tangent.

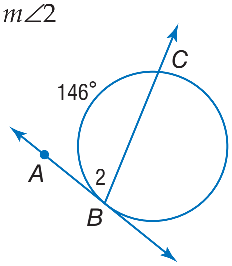
1. $m\angle 1$



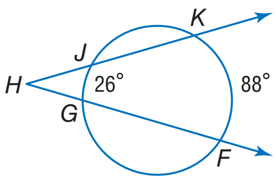
2. $m\widehat{TS}$



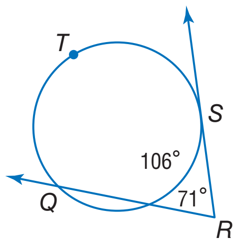
3. $m\angle 2$



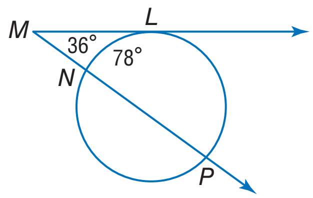
Examples 3–4 **4.** $m\angle H$



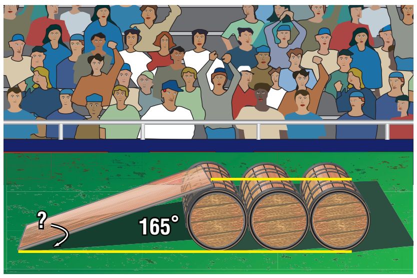
5. $m\widehat{QTS}$



6. $m\widehat{LP}$



7 **STUNTS** A ramp is attached to the first of several barrels that have been strapped together for a circus motorcycle stunt as shown. What is the measure of the angle the ramp makes with the ground?

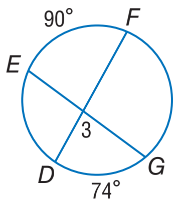


Practice and Problem Solving

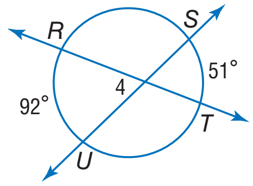
Extra Practice is on page R10.

Examples 1–2 Find each measure. Assume that segments that appear to be tangent are tangent.

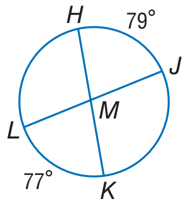
8. $m\angle 3$



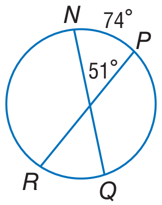
9. $m\angle 4$



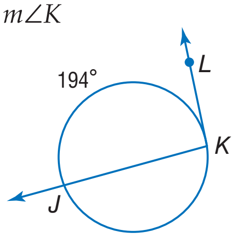
10. $m\angle JMK$



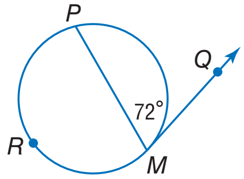
11 $m\widehat{RQ}$



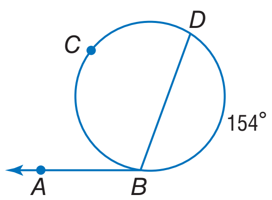
12. $m\angle K$



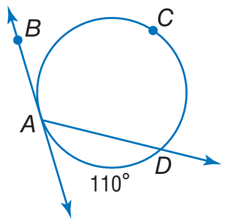
13. $m\widehat{PM}$



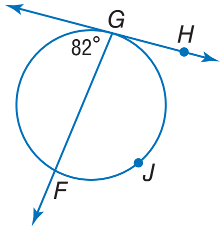
14. $m\angle ABD$



15. $m\angle DAB$



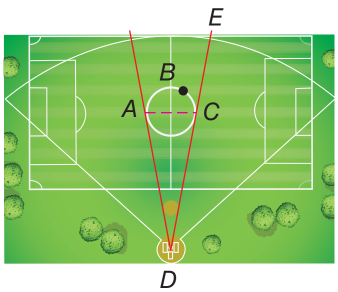
16. $m\widehat{GJF}$



17. SPORTS The multi-sport field shown includes a softball field and a soccer field. If $m\widehat{ABC} = 200$, find each measure.

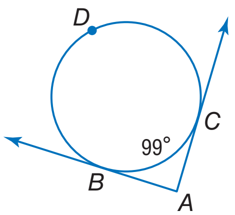
a. $m\angle ACE$

b. $m\angle ADC$

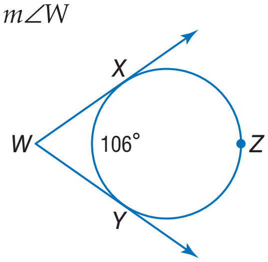


Examples 3–4 CCSS **STRUCTURE** Find each measure.

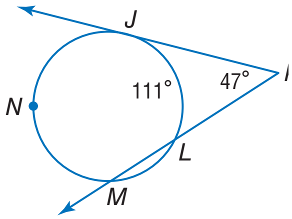
18. $m\angle A$



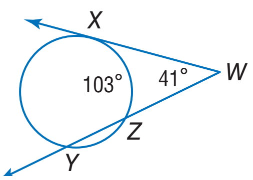
19. $m\angle W$



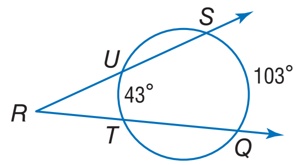
20. $m\widehat{JM}$



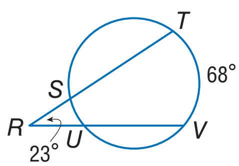
21. $m\widehat{XY}$



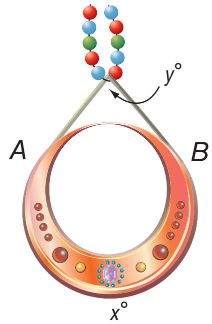
22. $m\angle R$



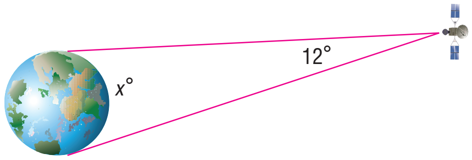
23. $m\widehat{SU}$



24. **JEWELRY** In the circular necklace shown, A and B are tangent points. If $x = 260$, what is y?

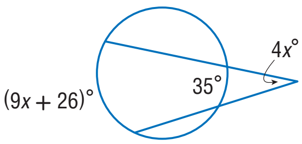


25. **SPACE** A satellite orbits above Earth's equator. Find x, the measure of the planet's arc, that is visible to the satellite.



ALGEBRA **Find the value of x.**

26.



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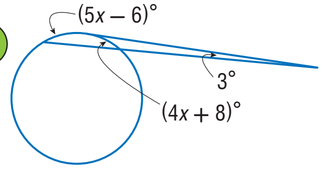

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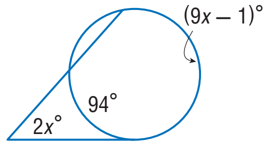
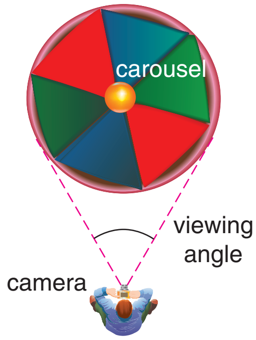


29. **PHOTOGRAPHY** A photographer frames a carousel in his camera shot as shown so that the lines of sight form tangents to the carousel.

a. If the camera's viewing angle is 35°, what is the arc measure of the carousel that appears in the shot?

b. If you want to capture an arc measure of 150° in the photograph, what viewing angle should be used?

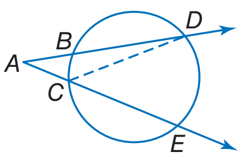


CCSS ARGUMENTS **For each case of Theorem 10.14, write a two-column proof.**

30. **Case 1**

Given: secants $\overrightarrow{AD}$ and $\overrightarrow{AE}$

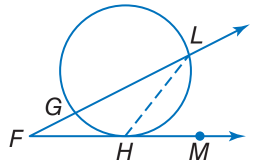
Prove: $m\angle A = \frac{1}{2}(m\widehat{DE} - m\widehat{BC})$



31. **Case 2**

Given: tangent $\overrightarrow{FM}$ and secant $\overrightarrow{FL}$

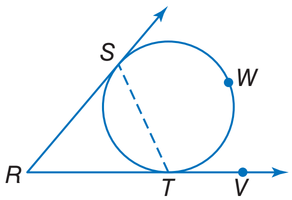
Prove: $m\angle F = \frac{1}{2}(m\widehat{LH} - m\widehat{GH})$



32. **Case 3**

Given: tangents $\overrightarrow{RS}$ and $\overrightarrow{RV}$

Prove: $m\angle R = \frac{1}{2}(m\widehat{SWT} - m\widehat{ST})$

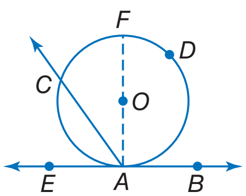


33. **PROOF** Write a paragraph proof of Theorem 10.13.

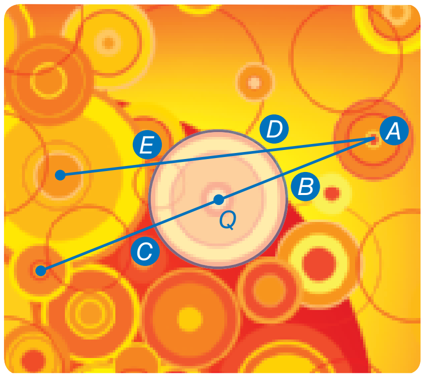
a. **Given:** $\overleftrightarrow{AB}$ is a tangent of $\odot O$.
$\overleftrightarrow{AC}$ is a secant of $\odot O$.
$\angle CAE$ is acute.

Prove: $m\angle CAE = \frac{1}{2}m\widehat{CA}$

b. Prove that if $\angle CAB$ is obtuse, $m\angle CAB = \frac{1}{2}m\widehat{CDA}$.

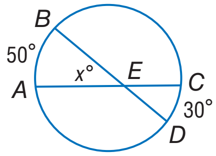


34. **WALLPAPER** In the wallpaper design shown, $\overline{BC}$ is a diameter of $\odot Q$. If $m\angle A = 26$ and $m\widehat{CE} = 67$, what is $m\widehat{DE}$?



35 **MULTIPLE REPRESENTATIONS** In this problem, you will explore the relationship between Theorems 10.12 and 10.6.

a. **Geometric** Copy the figure shown. Then draw three successive figures in which the position of point D moves closer to point C, but points A, B, and C remain fixed.



b. **Tabular** Estimate the measure of $\widehat{CD}$ for each successive circle, recording the measures of $\widehat{AB}$ and $\widehat{CD}$ in a table. Then calculate and record the value of x for each circle.

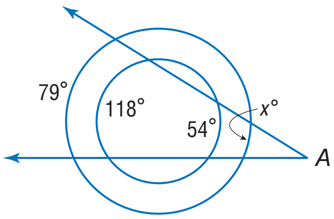
c. **Verbal** Describe the relationship between $m\widehat{AB}$ and the value of x as $m\widehat{CD}$ approaches zero. What type of angle does $\angle AEB$ become when $m\widehat{CD} = 0$?

d. **Analytical** Write an algebraic proof to show the relationship between Theorems 10.12 and 10.6 described in part **c**.

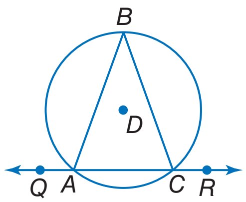
H.O.T. Problems Use Higher-Order Thinking Skills

36. **WRITING IN MATH** Explain how to find the measure of an angle formed by a secant and a tangent that intersect outside a circle.

37. **CHALLENGE** The circles below are concentric. What is x?



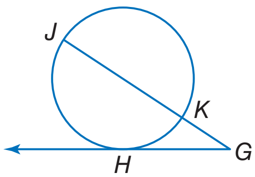
38. **REASONING** Isosceles $\triangle ABC$ is inscribed in $\odot D$. What can you conclude about $m\widehat{AB}$ and $m\widehat{BC}$? Explain.



39. **CCSS** **ARGUMENTS** In the figure, $\overline{JK}$ is a diameter and $\overrightarrow{GH}$ is a tangent.

a. Describe the range of possible values for $m\angle G$. Explain.

b. If $m\angle G = 34$, find the measures of minor arcs HJ and KH. Explain.

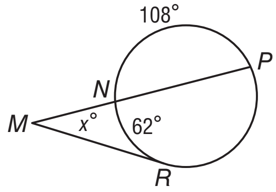


40. **OPEN ENDED** Draw a circle and two tangents that intersect outside the circle. Use a protractor to measure the angle that is formed. Find the measures of the minor and major arcs formed. Explain your reasoning.

41. **WRITING IN MATH** A circle is inscribed within $\triangle PQR$. If $m\angle P = 50$ and $m\angle Q = 60$, describe how to find the measures of the three minor arcs formed by the points of tangency.

Standardized Test Practice

42. What is the value of x if $m\widehat{NR} = 62$ and $m\widehat{NP} = 108$?

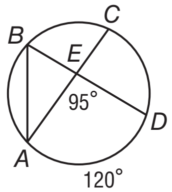


A 23° **C** 64°
B 31° **D** 128°

43. ALGEBRA Points $A(-4, 8)$ and $B(6, 2)$ are both on circle C, and $\overline{AB}$ is a diameter. What are the coordinates of C?

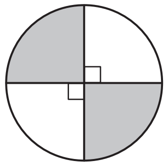
F (2, 10) **H** (5, −3)
G (10, −6) **J** (1, 5)

44. GRIDDED RESPONSE If $m\angle AED = 95$ and $m\widehat{AD} = 120$, what is $m\angle BAC$?



45. SAT/ACT If the circumference of the circle below is 16π units, what is the total area of the shaded regions?

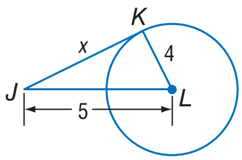
A 64π units^2 **D** 8π units^2
B 32π units^2 **E** 2π units^2
C 12π units^2



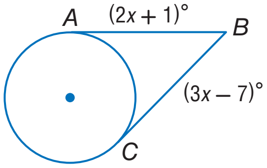
Spiral Review

Find x. Assume that segments that appear to be tangent are tangent. (Lesson 10-5)

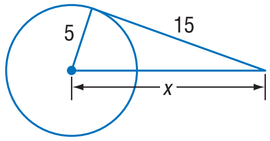
46.



47.



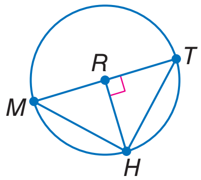
48.



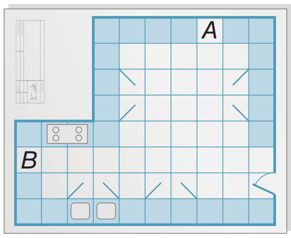
49. PROOF Write a two-column proof. (Lesson 10-4)

Given: $\widehat{MHT}$ is a semicircle; $\overline{RH} \perp \overline{TM}$.

Prove: $\frac{TR}{RH} = \frac{TH}{HM}$



50. REMODELING The diagram at the right shows the floor plan of Trent's kitchen. Each square on the diagram represents a 3-foot by 3-foot area. While remodeling his kitchen, Trent moved his refrigerator from square A to square B. Describe one possible combination of transformations that could be used to make this move. (Lesson 9-4)



COORDINATE GEOMETRY Find the measure of each angle to the nearest tenth of a degree by using the Distance Formula and an inverse trigonometric ratio. (Lesson 8-4)

51. $\angle C$ in triangle BCD with vertices $B(-1, -5)$, $C(-6, -5)$, and $D(-1, 2)$

52. $\angle X$ in right triangle XYZ with vertices $X(2, 2)$, $Y(2, -2)$, and $Z(7, -2)$

Skills Review

Solve each equation.

53. $x^2 + 13x = -36$

54. $x^2 - 6x = -9$

55. $3x^2 + 15x = 0$

56. $28 = x^2 + 3x$

57. $x^2 + 12x + 36 = 0$

58. $x^2 + 5x = -\frac{25}{4}$