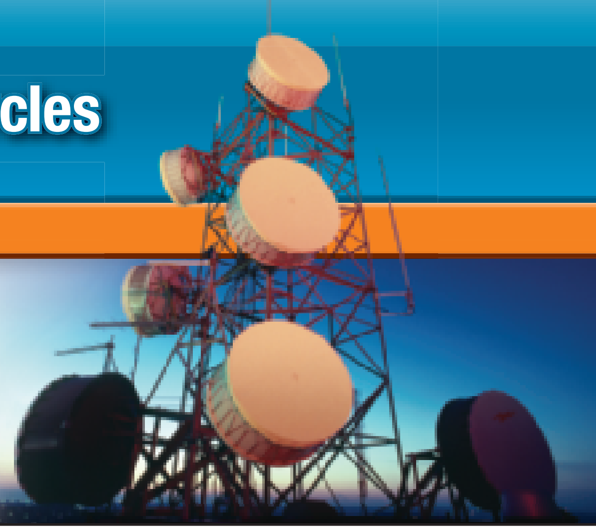
LESSON

10-8 Equations of Circles



Then

- You wrote equations of lines using information about their graphs.

Now

1. Write the equation of a circle.
2. Graph a circle on the coordinate plane.

Why?

- Telecommunications towers emit radio signals that are used to transmit cellular calls. Each tower covers a circular area, and towers are arranged so that a signal is available at any location in the coverage area.

NewVocabulary
compound locus

Common Core State Standards

Content Standards

G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Mathematical Practices

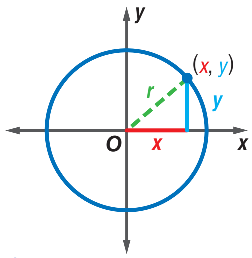
2 Reason abstractly and quantitatively.

7 Look for and make use of structure.

1 Equation of a Circle

Since all points on a circle are equidistant from the center, you can find an equation of a circle by using the Distance Formula.

Let (x, y) represent a point on a circle centered at the origin. Using the Pythagorean Theorem, $x^2 + y^2 = r^2$.



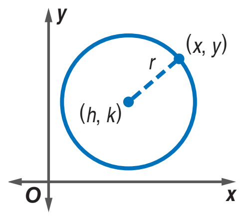
Now suppose that the center is not at the origin, but at the point (h, k). You can use the Distance Formula to develop an equation for the circle.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$r = \sqrt{(x - h)^2 + (y - k)^2}$	$d = r, (x_1, y_1) = (h, k), (x_2, y_2) = (x, y)$
$r^2 = (x - h)^2 + (y - k)^2$	Square each side.

KeyConcept Equation of a Circle in Standard Form

The standard form of the equation of a circle with center at (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

The standard form of the equation of a circle is also called the *center-radius* form.



Example 1 Write an Equation Using the Center and Radius

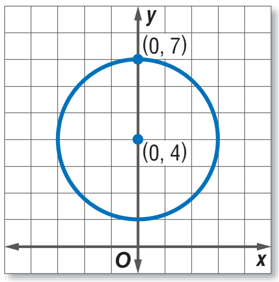
Write the equation of each circle.

a. center at (1, −8), radius 7

$(x - h)^2 + (y - k)^2 = r^2$	Equation of a circle
$(x - 1)^2 + [y - (-8)]^2 = 7^2$	$(h, k) = (1, -8), r = 7$
$(x - 1)^2 + (y + 8)^2 = 49$	Simplify.

b. the circle graphed at the right

The center is at (0, 4) and the radius is 3.



$(x - h)^2 + (y - k)^2 = r^2$	Equation of a circle
$(x - 0)^2 + (y - 4)^2 = 3^2$	$(h, k) = (0, 4), r = 3$
$x^2 + (y - 4)^2 = 9$	Simplify.

GuidedPractice

1A. center at origin, radius $\sqrt{10}$

1B. center at (4, −1), diameter 8

Greg Pease/Photographer's Choice/Getty Images

Example 2 Write an Equation Using the Center and a Point

Write the equation of the circle with center at (−2, 4), that passes through (−6, 7).

Step 1 Find the distance between the points to determine the radius.

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ Distance Formula

$$= \sqrt{[-6 - (-2)]^2 + (7 - 4)^2}$$ $(x_1, y_1) = (-2, 4)$ and $(x_2, y_2) = (-6, 7)$

$$= \sqrt{25} \text{ or } 5$$ Simplify.

Step 2 Write the equation using $h = -2$, $k = 4$, and $r = 5$.

$$(x - h)^2 + (y - k)^2 = r^2$$ Equation of a circle

$$[x - (-2)]^2 + (y - 4)^2 = 5^2$$ $h = -2$, $k = 4$, and $r = 5$

$$(x + 2)^2 + (y - 4)^2 = 25$$ Simplify.

GuidedPractice

2. Write the equation of the circle with center at (−3, −5) that passes through (0, 0).

2 Graph Circles

Graph Circles You can use the equation of a circle to graph it on a coordinate plane. To do so, you may need to write the equation in standard form first.

Example 3 Graph a Circle

The equation of a circle is $x^2 + y^2 - 8x + 2y = -8$. State the coordinates of the center and the measure of the radius. Then graph the equation.

Write the equation in standard form by completing the square.

$$x^2 + y^2 - 8x + 2y = -8$$ Original equation

$$x^2 - 8x + y^2 + 2y = -8$$ Isolate and group like terms.

$$x^2 - 8x + 16 + y^2 + 2y + 1 = -8 + 16 + 1$$ Complete the squares.

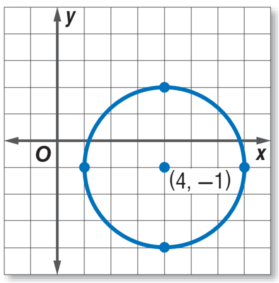
$$(x - 4)^2 + (y + 1)^2 = 9$$ Factor and simplify.

$$(x - 4)^2 + [y - (-1)]^2 = 3^2$$ Write +1 as − (−1) and 9 as 3^2.

With the equation now in standard form, you can identify h, k, and r.

$$(x - 4)^2 + [y - (-1)]^2 = 3^2$$
$$\uparrow \quad \uparrow \quad \uparrow$$
$$(x - h)^2 + (y - k)^2 = r^2$$

So, $h = 4$, $k = -1$, and $r = 3$. The center is at (4, −1), and the radius is 3. Plot the center and four points that are 3 units from this point. Sketch the circle through these four points.



StudyTip

Completing the Square To complete the square for any quadratic expression of the form $x^2 + bx$, follow these steps.

Step 1 Find one half of b.

Step 2 Square the result in Step 1.

Step 3 Add the result of Step 2 to $x^2 + bx$.

GuidedPractice

For each circle with the given equation, state the coordinates of the center and the measure of the radius. Then graph the equation.

3A. $x^2 + y^2 - 4 = 0$

3B. $x^2 + y^2 + 8x - 14y + 40 = 0$

Real-WorldLink

About 1000 tornadoes are reported across the United States each year. The most violent tornadoes have wind speeds of 250 mph or more. Damage paths can be a mile wide and 50 miles long.

Source: National Oceanic & Atmospheric Administration

Real-World Example 4 Use Three Points to Write an Equation

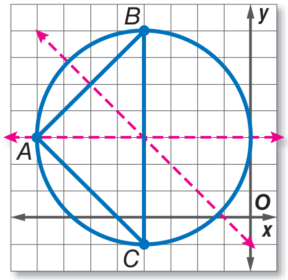
PT

TORNADOES Three tornado sirens are placed strategically on a circle around a town so they can be heard by all. Write the equation of the circle on which they are placed if the coordinates of the sirens are $A(-8, 3)$, $B(-4, 7)$, and $C(-4, -1)$.

Understand You are given three points that lie on a circle.

Plan Graph $\triangle ABC$. Construct the perpendicular bisectors of two sides to locate the center of the circle. Then find the radius.

Use the center and radius to write an equation.



Solve The center appears to be at $(-4, 3)$. The radius is 4. Write an equation.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$[x - (-4)]^2 + (y - 3)^2 = 4^2$$

$$(x + 4)^2 + (y - 3)^2 = 16$$

Check Verify the center by finding the equations of the two bisectors and solving the system of equations. Verify the radius by finding the distance between the center and another point on the circle. ✔

GuidedPractice

4. Write an equation of a circle that contains $R(1, 2)$, $S(-3, 4)$, and $T(-5, 0)$.

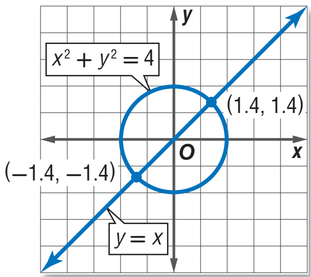
A line can intersect a circle in at most two points. You can find the point(s) of intersection between a circle and a line by applying techniques used to find the intersection between two lines and techniques used to solve quadratic equations.

Example 5 Intersections with Circles

PT

Find the point(s) of intersection between $x^2 + y^2 = 4$ and $y = x$.

Graph these equations on the same coordinate plane. The points of intersection are solutions of both equations. You can estimate these points on the graph to be at about $(-1.4, -1.4)$ and $(1.4, 1.4)$. Use substitution to find the coordinates of these points algebraically.



$x^2 + y^2 = 4$	Equation of circle
$x^2 + x^2 = 4$	Since $y = x$, substitute x for y.
$2x^2 = 4$	Simplify.
$x^2 = 2$	Divide each side by 2.
$x = \pm\sqrt{2}$	Take the square root of each side.

So $x = \sqrt{2}$ or $x = -\sqrt{2}$. Use the equation $y = x$ to find the corresponding y-values.

$y = x$	Equation of line	$y = x$
$y = \sqrt{2}$	$x = \sqrt{2}$ or $x = -\sqrt{2}$	$y = -\sqrt{2}$

The points of intersection are located at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ or at about $(-1.4, -1.4)$ and $(1.4, 1.4)$. Check these solutions in both of the original equations.

StudyTip

Quadratic Techniques In addition to taking square roots, other quadratic techniques that you may need to apply in order to solve equations of the form $ax^2 + bx + c = 0$ include completing the square, factoring, and the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

GuidedPractice

5. Find the point(s) of intersection between $x^2 + y^2 = 8$ and $y = -x$.

Warren Faidley/CORBIS

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Examples 1–2 **Write the equation of each circle.**

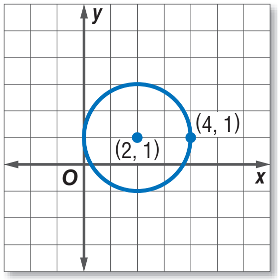
1. center at (9, 0), radius 5

2. center at (3, 1), diameter 14

3 center at origin, passes through (2, 2)

4. center at (−5, 3), passes through (1, −4)

5.



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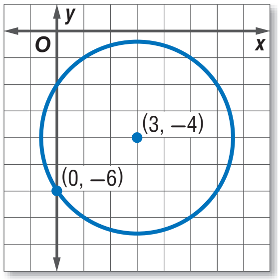


Example 3 **For each circle with the given equation, state the coordinates of the center and the measure of the radius. Then graph the equation.**

7. $x^2 - 6x + y^2 + 4y = 3$

8. $x^2 + (y + 1)^2 = 4$

Example 4 **9.** **RADIOS** Three radio towers are modeled by the points $R(4, 5)$, $S(8, 1)$, and $T(-4, 1)$. Determine the location of another tower equidistant from all three towers, and write an equation for the circle.

10. **COMMUNICATION** Three cell phone towers can be modeled by the points $X(6, 0)$, $Y(8, 4)$, and $Z(3, 9)$. Determine the location of another cell phone tower equidistant from the other three, and write an equation for the circle.

Example 5 **Find the point(s) of intersection, if any, between each circle and line with the equations given.**

11. $(x - 1)^2 + y^2 = 4$
$y = x + 1$

12. $(x - 2)^2 + (y + 3)^2 = 18$
$y = -2x - 2$

Practice and Problem Solving

Extra Practice is on page R10.

Examples 1–2 CCSS **STRUCTURE** **Write the equation of each circle.**

13. center at origin, radius 4

14. center at (6, 1), radius 7

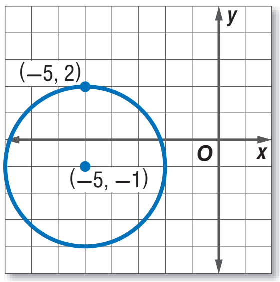
15. center at (−2, 0), diameter 16

16. center at (8, −9), radius $\sqrt{11}$

17. center at (−3, 6), passes through (0, 6)

18. center at (1, −2), passes through (3, −4)

19.



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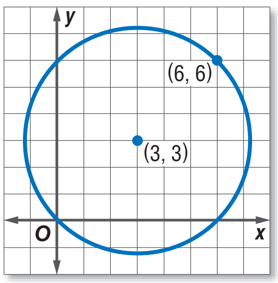


21 **WEATHER** A Doppler radar screen shows concentric rings around a storm. If the center of the radar screen is the origin and each ring is 15 miles farther from the center, what is the equation of the third ring?

22. **GARDENING** A sprinkler waters a circular area that has a diameter of 10 feet. The sprinkler is located 20 feet north of the house. If the house is located at the origin, what is the equation for the circle of area that is watered?

Example 3 **For each circle with the given equation, state the coordinates of the center and the measure of the radius. Then graph the equation.**

23. $x^2 + y^2 = 36$

24. $x^2 + y^2 - 4x - 2y = -1$

25. $x^2 + y^2 + 8x - 4y = -4$

26. $x^2 + y^2 - 16x = 0$

Example 4 **Write an equation of a circle that contains each set of points. Then graph the circle.**

27. $A(1, 6), B(5, 6), C(5, 0)$

28. $F(3, -3), G(3, 1), H(7, 1)$

Example 5 **Find the point(s) of intersection, if any, between each circle and line with the equations given.**

29. $x^2 + y^2 = 5$, $y = \frac{1}{2}x$

30. $x^2 + y^2 = 2$, $y = -x + 2$

31. $x^2 + (y + 2)^2 = 8$, $y = x - 2$

32. $(x + 3)^2 + y^2 = 25$, $y = -3x$

33. $x^2 + y^2 = 5$, $y = 3x$

34. $(x - 1)^2 + (y - 3)^2 = 4$, $y = -x$

Write the equation of each circle.

35. a circle with a diameter having endpoints at (0, 4) and (6, −4)

36. a circle with $d = 22$ and a center translated 13 units left and 6 units up from the origin

37. CCSS **MODELING** Different-sized engines will launch model rockets to different altitudes. The higher a rocket goes, the larger the circle of possible landing sites becomes. Under normal wind conditions, the landing radius is three times the altitude of the rocket.

a. Write the equation of the landing circle for a rocket that travels 300 feet in the air.

b. What would be the radius of the landing circle for a rocket that travels 1000 feet in the air? Assume the center of the circle is at the origin.

38. **SKYDIVING** Three of the skydivers in the circular formation shown have approximate coordinates of $G(13, -2)$, $H(-1, -2)$, and $J(6, -9)$.

a. What are the approximate coordinates of the center skydiver?

b. If each unit represents 1 foot, what is the diameter of the skydiving formation?

39. **DELIVERY** Pizza and Subs offers free delivery within 6 miles of the restaurant. The restaurant is located 4 miles west and 5 miles north of Consuela's house.

a. Write and graph an equation to represent this situation if Consuela's house is at the origin of the coordinate system.

b. Can Consuela get free delivery if she orders pizza from Pizza and Subs? Explain.

40. **INTERSECTIONS OF CIRCLES** Graph $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$ on the same coordinate plane.

a. Estimate the point(s) of intersection between the two circles.

b. Solve $x^2 + y^2 = 4$ for y.

c. Substitute the value you found in part **b** into $(x - 2)^2 + y^2 = 4$ and solve for x.

d. Substitute the value you found in part **c** into $x^2 + y^2 = 4$ and solve for y.

e. Use your answers to parts **c** and **d** to write the coordinates of the points of intersection. Compare these coordinates to your estimate from part **a**.

f. Verify that the point(s) you found in part **d** lie on both circles.

41 Prove or disprove that the point $(1, 2\sqrt{2})$ lies on a circle centered at the origin and containing the point $(0, -3)$.

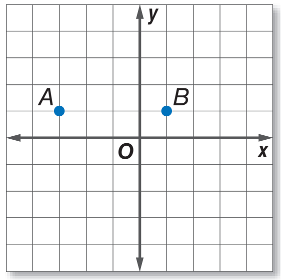
42. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate a compound locus for a pair of points. A **compound locus** satisfies more than one distinct set of conditions.

a. Tabular Choose two points A and B in the coordinate plane. Locate 5 coordinates from the locus of points equidistant from A and B.

b. Graphical Represent this same locus of points by using a graph.

c. Verbal Describe the locus of all points equidistant from a pair of points.

d. Graphical Using your graph from part **b**, determine and graph the locus of all points in a plane that are a distance of AB from B.

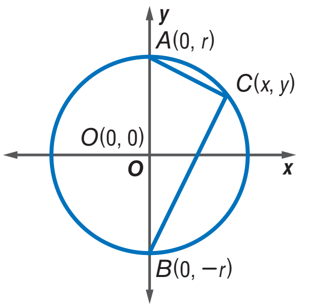


e. Verbal Describe the locus of all points in a plane equidistant from a single point. Then describe the locus of all points that are both equidistant from A and B and are a distance of AB from B. Describe the graph of the compound locus.

43. A circle with a diameter of 12 has its center in the second quadrant. The lines $y = -4$ and $x = 1$ are tangent to the circle. Write an equation of the circle.

H.O.T. Problems Use Higher-Order Thinking Skills

44. CHALLENGE Write a coordinate proof to show that if an inscribed angle intercepts the diameter of a circle, as shown, the angle is a right angle.



45. CCSS REASONING A circle has the equation $(x - 5)^2 + (y + 7)^2 = 16$. If the center of the circle is shifted 3 units right and 9 units up, what would be the equation of the new circle? Explain your reasoning.

46. OPEN ENDED Graph three noncollinear points and connect them to form a triangle. Then construct the circle that circumscribes it.

47. WRITING IN MATH Seven new radio stations must be assigned broadcast frequencies. The stations are located at $A(9, 2)$, $B(8, 4)$, $C(8, 1)$, $D(6, 3)$, $E(4, 0)$, $F(3, 6)$, and $G(4, 5)$, where 1 unit = 50 miles.

a. If stations that are more than 200 miles apart can share the same frequency, what is the least number of frequencies that can be assigned to these stations?

b. Describe two different beginning approaches to solving this problem.

c. Choose an approach, solve the problem, and explain your reasoning.

CHALLENGE Find the coordinates of point P on $\overrightarrow{AB}$ that partitions the segment into the given ratio AP to PB.

48. $A(0, 0)$, $B(3, 4)$, 2 to 3

49. $A(0, 0)$, $B(-8, 6)$, 4 to 1

50. WRITING IN MATH Describe how the equation for a circle changes if the circle is translated a units to the right and b units down.

Standardized Test Practice

51. Which of the following is the equation of a circle with center (6, 5) that passes through (2, 8)?

A $(x-6)^2+(y-5)^2=5^2$
B $(x-5)^2+(y-6)^2=7^2$
C $(x+6)^2+(y+5)^2=5^2$
D $(x-2)^2+(y-8)^2=7^2$

52. ALGEBRA What are the solutions of $n^2-4n=21$?

F 3, 7 **H** −3, 7
G 3, −7 **J** −3, −7

53. SHORT RESPONSE Solve: $5(x-4)=16$.

Step 1: $5x-4=16$
Step 2: $5x=20$
Step 3: $x=4$

Which is the first incorrect step in the solution shown above?

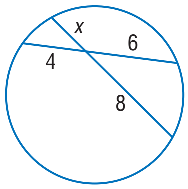
54. SAT/ACT The center of ⊙F is at (−4, 0) and has a radius of 4. Which point lies on ⊙F?

A (4, 0) **D** (−4, 4)
B (0, 4) **E** (0, 8)
C (4, 3)

Spiral Review

Find x. (Lesson 10-7)

55.



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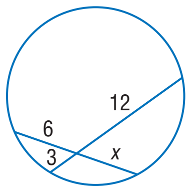

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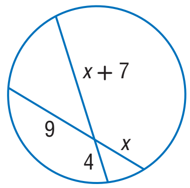
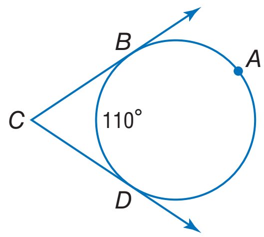
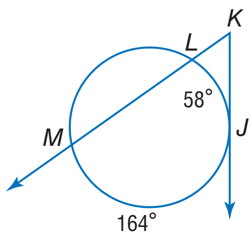


Find each measure. (Lesson 10-6)

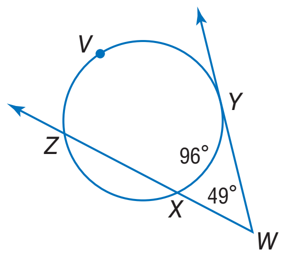
58. $m\angle C$



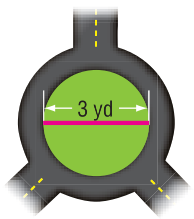
59. $m\angle K$



60. $m\widehat{YXZ}$



61. STREETS The neighborhood where Vincent lives has roundabouts where certain streets meet. If Vincent rides his bike once around the very edge of the grassy circle, how many feet will he have ridden? (Lesson 10-1)



Skills Review

Find the perimeter and area of each figure.

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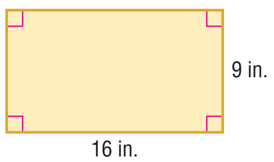

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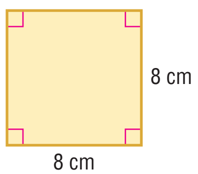

64.