LESSON 2-6

Algebraic Proof

Then

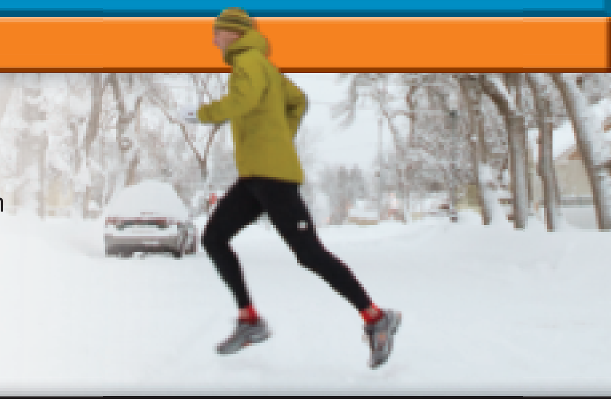
- You used postulates about points, lines, and planes to write paragraph proofs.

Now

1. Use algebra to write two-column proofs.
2. Use properties of equality to write geometric proofs.

Why?

- The Fahrenheit scale sets the freezing and boiling points of water at 32° and 212°, respectively, while the Celsius scale sets them at 0° and 100°. You can use an algebraic proof to show that if these scales are related by the formula $C = \frac{5}{9}(F - 32)$, then they are also related by the formula $F = \frac{9}{5}C + 32$.



NewVocabulary
algebraic proof
two-column proof
formal proof

Common Core State Standards

Content Standards
Preparation for G.CO.9
Prove theorems about lines and angles.

Mathematical Practices
3 Construct viable arguments and critique the reasoning of others.

1 Algebraic Proof

Algebra is a system with sets of numbers, operations, and properties that allow you to perform algebraic operations. The following table summarizes several properties of real numbers that you studied in algebra.

KeyConcept Properties of Real Numbers

The following properties are true for any real numbers *a*, *b*, and *c*.

Addition Property of Equality	If $a = b$, then $a + c = b + c$.
Subtraction Property of Equality	If $a = b$, then $a - c = b - c$.
Multiplication Property of Equality	If $a = b$, then $a \cdot c = b \cdot c$.
Division Property of Equality	If $a = b$ and $c \neq 0$, then, $\frac{a}{c} = \frac{b}{c}$.
Reflexive Property of Equality	$a = a$
Symmetric Property of Equality	If $a = b$, then $b = a$.
Transitive Property of Equality	If $a = b$ and $b = c$, then $a = c$.
Substitution Property of Equality	If $a = b$, then a may be replaced by b in any equation or expression.
Distributive Property	$a(b + c) = ab + ac$

An **algebraic proof** is a proof that is made up of a series of algebraic statements. The properties of equality provide justification for many statements in algebraic proofs.

PT

Example 1 Justify Each Step When Solving an Equation

Prove that if $-5(x + 4) = 70$, then $x = -18$. Write a justification for each step.

$-5(x + 4) = 70$	Original equation or Given
$-5x + (-5)4 = 70$	Distributive Property
$-5x - 20 = 70$	Substitution Property of Equality
$-5x - 20 + 20 = 70 + 20$	Addition Property of Equality
$-5x = 90$	Substitution Property of Equality
$\frac{-5x}{-5} = \frac{90}{-5}$	Division Property of Equality
$x = -18$	Substitution Property of Equality

Heath Korvola/Digital Vision/Getty Images

GuidedPractice

State the property that justifies each statement.

1A. If $4 + (-5) = -1$, then $x + 4 + (-5) = x - 1$.

1B. If $5 = y$, then $y = 5$.

1C. Prove that if $2x - 13 = -5$, then $x = 4$. Write a justification for each step.

Example 1 is a proof of the conditional statement *If* $-5(x + 4) = 70$, *then* $x = -18$. Notice that the column on the left is a step-by-step process that leads to a solution. The column on the right contains the reason for each statement.

StudyTip

CCSS Arguments An *algorithm* is a series of steps for carrying out a procedure or solving a problem. Proofs can be considered a type of algorithm because they go step by step.

In geometry, a similar format is used to prove conjectures and theorems. A **two-column proof** or **formal proof** contains *statements* and *reasons* organized in two columns.

Real-World Example 2 Write an Algebraic Proof

SCIENCE If the formula to convert a Fahrenheit temperature to a Celsius temperature is $C = \frac{5}{9}(F - 32)$, then the formula to convert a Celsius temperature to a Fahrenheit temperature is $F = \frac{9}{5}C + 32$. Write a two-column proof to verify this conjecture.

Begin by stating what is given and what you are to prove.

Given: $C = \frac{5}{9}(F - 32)$

Prove: $F = \frac{9}{5}C + 32$

Proof:

Statements	Reasons
1. $C = \frac{5}{9}(F - 32)$	**1.** Given
2. $\frac{9}{5}C = \frac{9}{5} \cdot \frac{5}{9}(F - 32)$	**2.** Multiplication Property of Equality
3. $\frac{9}{5}C = F - 32$	**3.** Substitution Property of Equality
4. $\frac{9}{5}C + 32 = F - 32 + 32$	**4.** Addition Property of Equality
5. $\frac{9}{5}C + 32 = F$	**5.** Substitution Property of Equality
6. $F = \frac{9}{5}C + 32$	**6.** Symmetric Property of Equality

StudyTip

Mental Math If your teacher permits you to do so, some steps may be eliminated by performing mental calculations. For example, steps 2 and 4 in Example 2 could be omitted. Then the reason for statement 3 would be Multiplication Property of Equality and the reason for statement 5 would be Addition Property of Equality.

GuidedPractice

Write a two-column proof to verify that each conjecture is true.

2A. If $\frac{5x + 1}{2} - 8 = 0$, then $x = 3$.

2B. **PHYSICS** If the distance d moved by an object with initial velocity u and final velocity v in time t is given by $d = t \cdot \frac{u + v}{2}$, then $u = \frac{2d}{t} - v$.

2 Geometric Proof Since geometry also uses variables, numbers, and operations, many of the properties of equality used in algebra are also true in geometry. For example, segment measures and angle measures are real numbers, so properties from algebra can be used to discuss their relationships as shown in the table below.

Property	Segments	Angles
Reflexive	$AB = AB$	$m\angle 1 = m\angle 1$
Symmetric	If $AB = CD$, then $CD = AB$.	If $m\angle 1 = m\angle 2$, then $m\angle 2 = m\angle 1$.
Transitive	If $AB = CD$ and $CD = EF$, then $AB = EF$.	If $m\angle 1 = m\angle 2$ and $m\angle 2 = m\angle 3$, then $m\angle 1 = m\angle 3$.

StudyTip

Commutative and Associative Properties Throughout this text we shall assume that if a, b, and c are real numbers, then the following properties are true.

Commutative Property of Addition
$a + b = b + a$

Commutative Property of Multiplication
$a \cdot b = b \cdot a$

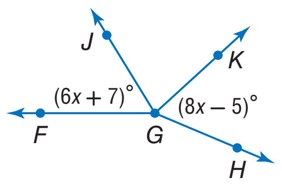
Associative Property of Addition
$(a + b) + c = a + (b + c)$

Associative Property of Multiplication
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

These properties can be used to write geometric proofs.

Example 3 Write a Geometric Proof

If $\angle FGJ \cong \angle JGK$ and $\angle JGK \cong \angle KGH$, then $x = 6$. Write a two-column proof to verify this conjecture.



Given: $\angle FGJ \cong \angle JGK$, $\angle JGK \cong \angle KGH$, $m\angle FGJ = 6x + 7$, $m\angle KGH = 8x - 5$

Prove: $x = 6$

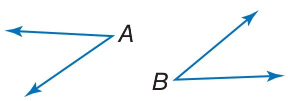
Proof:

Statements	Reasons
1. $m\angle FGH = 6x + 7$, $m\angle KGH = 8x - 5$ $\angle FGJ \cong \angle JGK$; $\angle JGK \cong \angle KGH$	**1.** Given
	2. Definition of congruent angles
2. $m\angle FGJ = m\angle JGK$; $m\angle JGK = m\angle KGH$	
3. $m\angle FGJ = m\angle KGH$	**3.** Transitive Property of Equality
4. $6x + 7 = 8x - 5$	**4.** Substitution Property of Equality
5. $6x + 7 + 5 = 8x - 5 + 5$	**5.** Addition Property of Equality
6. $6x + 12 = 8x$	**6.** Substitution Property of Equality
7. $6x + 12 - 6x = 8x - 6x$	**7.** Subtraction Property of Equality
8. $12 = 2x$	**8.** Substitution Property of Equality
9. $\frac{12}{2} = \frac{2x}{2}$	**9.** Division Property of Equality
10. $6 = x$	**10.** Substitution Property of Equality
11. $x = 6$	**11.** Symmetric Property of Equality

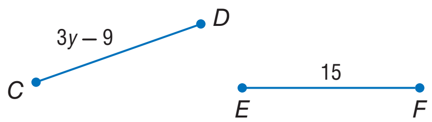
GuidedPractice

Write a two-column proof to verify each conjecture.

3A. If $\angle A \cong \angle B$ and $m\angle A = 37$, then $m\angle B = 37$.



3B. If $\overline{CD} \cong \overline{EF}$, then $y = 8$.



Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Example 1 **State the property that justifies each statement.**

1. If $m\angle 1 = m\angle 2$ and $m\angle 2 = m\angle 3$, then $m\angle 1 = m\angle 3$.

2. $XY = XY$

3. If $5 = x$, then $x = 5$.

4. If $2x + 5 = 11$, then $2x = 6$.

Example 2 **5.** Complete the following proof.

Given: $\frac{y+2}{3} = 3$

Prove: $y = 7$

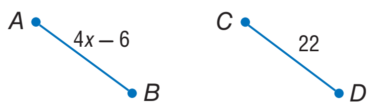
Proof:

Statements	Reasons
a. ___?___	**a.** Given
b. $3\left(\frac{y+2}{3}\right) = 3(3)$	**b.** ___?___
c. ___?___	**c.** ___?___
d. $y = 7$	**d.** Subtraction Property

Examples 2–3 **PROOF** **Write a two-column proof to verify each conjecture.**

6. If $-4(x - 3) + 5x = 24$, then $x = 12$.

7. If $\overline{AB} \cong \overline{CD}$, then $x = 7$.



8. CCSS **ARGUMENTS** Mai-Lin measures her heart rate whenever she exercises and tries to make sure that she is staying in her target heart rate zone. The American Heart Association suggests a target heart rate of $T = 0.75(220 - a)$, where T is a person's target heart rate and a is his or her age.

a. Prove that given a person's target heart rate, you can calculate his or her age using the formula $a = 220 - \frac{T}{0.75}$.

b. If Mai-Lin's target heart rate is 153, then how old is she? What property justifies your calculation?

Practice and Problem Solving

Extra Practice is on page R2.

Example 1 **State the property that justifies each statement.**

9. If $a + 10 = 20$, then $a = 10$.

10. If $\frac{x}{3} = -15$, then $x = -45$.

11. If $4x - 5 = x + 12$, then $4x = x + 17$.

12. If $\frac{1}{5}BC = \frac{1}{5}DE$, then $BC = DE$.

State the property that justifies each statement.

13. If $5(x + 7) = -3$, then $5x + 35 = -3$.

14. If $m\angle 1 = 25$ and $m\angle 2 = 25$, then $m\angle 1 = m\angle 2$.

15. If $AB = BC$ and $BC = CD$, then $AB = CD$.

16. If $3\left(x - \frac{2}{3}\right) = 4$, then $3x - 2 = 4$.

Example 2 CCSS **ARGUMENTS** **Complete each proof.**

17. Given: $\frac{8 - 3x}{4} = 32$

Prove: $x = -40$

Proof:

Statements	Reasons
a. $\frac{8 - 3x}{4} = 32$	**a.** Given
b. $4\left(\frac{8 - 3x}{4}\right) = 4(32)$	**b.** ___?___
c. $8 - 3x = 128$	**c.** ___?___
d. ___?___	**d.** Subtraction Property
e. $x = -40$	**e.** ___?___

18. Given: $\frac{1}{5}x + 3 = 2x - 24$

Prove: $x = 15$

Proof:

Statements	Reasons
a. ___?___	**a.** Given
b. ___?___	**b.** Multiplication Property
c. $x + 15 = 10x - 120$	**c.** ___?___
d. ___?___	**d.** Subtraction Property
e. $135 = 9x$	**e.** ___?___
f. ___?___	**f.** Division Property
g. ___?___	**g.** Symmetric Property

Example 3 **PROOF** **Write a two-column proof to verify each conjecture.**

19. If $-\frac{1}{3}n = 12$, then $n = -36$.

20. If $-3r + \frac{1}{2} = 4$, then $r = -\frac{7}{6}$.

21 **SCIENCE** Acceleration a in feet per second squared, distance traveled d in feet, velocity v in feet per second, and time t in seconds are related in the formula $d = vt + \frac{1}{2}at^2$.

a. Prove that if the values for distance, velocity, and time are known, then the acceleration of an object can be calculated using the formula $a = \frac{2d - 2vt}{t^2}$.

b. If an object travels 2850 feet in 30 seconds with an initial velocity of 50 feet per second, what is the acceleration of the object? What property justifies your calculation?

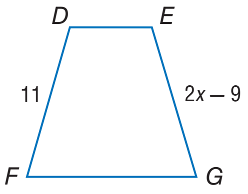
22. **CCSS ARGUMENTS** The Ideal Gas Law is given by the formula $PV = nRT$, where $P =$ pressure in atmospheres, $V =$ volume in liters, $n =$ the amount of gas in moles, R is a constant value, and $T =$ temperature in degrees Kelvin.

a. Prove that if the pressure, volume, and amount of the gas are known, then the formula $T = \frac{PV}{nR}$ gives the temperature of the gas.

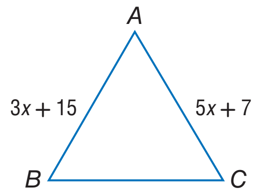
b. If you have 1 mole of oxygen with a volume of 25 liters at a pressure of 1 atmosphere, what is the temperature of the gas? The value of R is 0.0821. What property justifies your calculation?

PROOF **Write a two-column proof.**

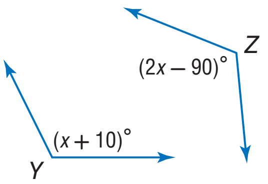
23. If $\overline{DF} \cong \overline{EG}$, then $x = 10$.



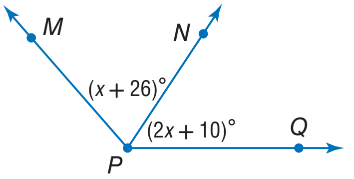
24. If $\overline{AB} \cong \overline{AC}$, then $x = 4$.



25. If $\angle Y \cong \angle Z$, then $x = 100$.



26. If $\angle MPN \cong \angle QPN$, then $x = 16$.

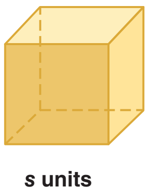


27. **ELECTRICITY** The voltage V of a circuit can be calculated using the formula $V = \frac{P}{I}$, where P is the power and I is the current of the circuit.

a. Write a proof to show that when the power is constant, the voltage is halved when the current is doubled.

b. Write a proof to show that when the current is constant, the voltage is doubled when the power is doubled.

28. **MULTIPLE REPRESENTATIONS** Consider a cube with a side length of s.



a. **Concrete** Sketch or build a model of cubes with side lengths of 2, 4, 8, and 16 units.

b. **Tabular** Find the volume of each cube. Organize your results into a table like the one shown.

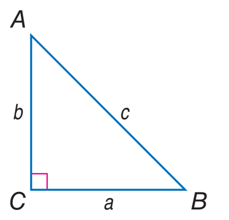
Side Lenth (s)	Volume (V)
2	
4	
8	
16	

c. **Verbal** Use your table to make a conjecture about the change in volume when the side length of a cube is doubled. Express your conjecture in words.

d. **Analytical** Write your conjecture as an algebraic equation.

e. **Logical** Write a proof of your conjecture. Be sure to write the *Given* and *Prove* statements at the beginning of your proof.

29. **PYTHAGOREAN THEOREM** The Pythagorean Theorem states that in a right triangle ABC, the sum of the squares of the measures of the lengths of the legs, a and b, equals the square of the measure of the hypotenuse c, or $a^2 + b^2 = c^2$. Write a two-column proof to verify that $a = \sqrt{c^2 - b^2}$. Use the Square Root Property of Equality, which states that if $a^2 = b^2$, then $a = \pm\sqrt{b^2}$.

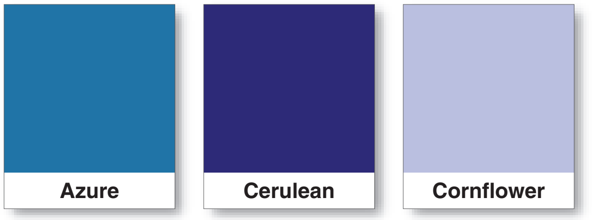


An *equivalence relation* is any relationship that satisfies the Reflexive, Symmetric, and Transitive Properties. For real numbers, equality is one type of equivalence relation. Determine whether each relation is an equivalence relation. Explain your reasoning.

30. "has the same birthday as," for the set of all human beings

31 "is taller than," for the set of all human beings

32. "is bluer than" for all the paint colors with blue in them



33. $\neq$, for the set of real numbers

34. $\geq$, for the set of real numbers

35. $\approx$, for the set of real numbers

H.O.T. Problems Use Higher-Order Thinking Skills

36. **OPEN ENDED** Give one real-world *example* and one real-world *non-example* of the Symmetric, Transitive, and Substitution properties.

37. CCSS **SENSE-MAKING** Point P is located on $\overline{AB}$. The length of $\overline{AP}$ is $2x + 3$, and the length of $\overline{PB}$ is $\frac{3x + 1}{2}$. Segment AB is 10.5 units long. Draw a diagram of this situation, and prove that point P is located two thirds of the way between point A and point B.

REASONING **Classify each statement below as *sometimes*, *always*, or *never* true. Explain your reasoning.**

38. If a and b are real numbers and $a + b = 0$, then $a = -b$.

39. If a and b are real numbers and $a^2 = b$, then $a = \sqrt{b}$.

40. **CHALLENGE** Ayana makes a conjecture that the sum of two odd integers is an even integer.

a. List information that supports this conjecture. Then explain why the information you listed does not prove that this conjecture is true.

b. Two odd integers can be represented by the expressions $2n - 1$ and $2m - 1$, where n and m are both integers. Give information that supports this statement.

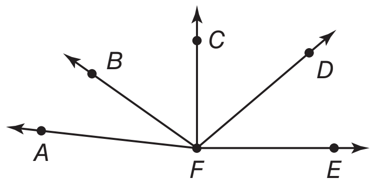
c. If a number is even, then it is a multiple of what number? Explain in words how you could use the expressions in part **a** and your answer to part **b** to prove Ayana's conjecture.

d. Write an algebraic proof that the sum of two odd integers is an even integer.

41. **WRITING IN MATH** Why is it useful to have different formats that can be used when writing a proof?

Standardized Test Practice

42. In the diagram, $m\angle CFE = 90$ and $\angle AFB \cong \angle CFD$. Which of the following conclusions does not have to be true?



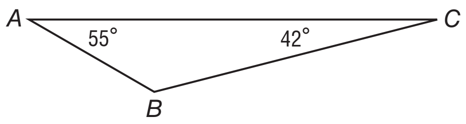
A $m\angle BFD = m\angle BFD$

B $\overline{BF}$ bisects $\angle AFD$.

C $m\angle CFD = m\angle AFB$

D $\angle CFE$ is a right angle.

43. SHORT RESPONSE Find the measure of $\angle B$ when $m\angle A = 55$ and $m\angle C = 42$.



44. ALGEBRA Kendra's walk-a-thon supporters have pledged \$30 plus \$7.50 for each mile she walks. Rebecca's supporters have pledged \$45 plus \$3.75 for each mile she walks. After how many miles will Kendra and Rebecca have raised the same amount of money?

F 10

G 8

H 5

J 4

45. SAT/ACT When 17 is added to $4m$, the result is $15z$. Which of the following equations represents the statement above?

A $17 + 15z = 4m$

B $(4m)(15z) = 17$

C $4m - 15z = 17$

D $17(4m) = 15z$

E $4m + 17 = 15z$

Spiral Review

Determine whether the following statements are *always*, *sometimes*, or *never* true. Explain. (Lesson 2-5)

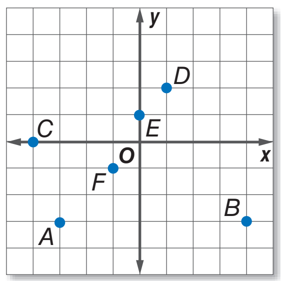
46. Four points will lie in one plane.

47. Two obtuse angles will be supplementary.

48. Planes $\mathcal{P}$ and $\mathcal{Q}$ intersect in line m. Line m lies in both plane $\mathcal{P}$ and plane $\mathcal{Q}$.

49. ADVERTISING An ad for Speedy Delivery Service says *When it has to be there fast, it has to be Speedy*. Catalina needs to send a package fast. Does it follow that she should use Speedy? Explain. (Lesson 2-4)

Write the ordered pair for each point shown. (Lesson 0-7)



50. A

51. B

52. C

53. D

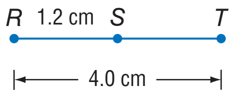
54. E

55. F

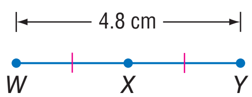
Skills Review

Find the measurement of each segment. Assume that each figure is not drawn to scale.

56. $\overline{ST}$



57. $\overline{WX}$



58. $\overline{BC}$