LESSON 3-6 Perpendiculars and Distance

Then

- You proved that two lines are parallel using angle relationships.

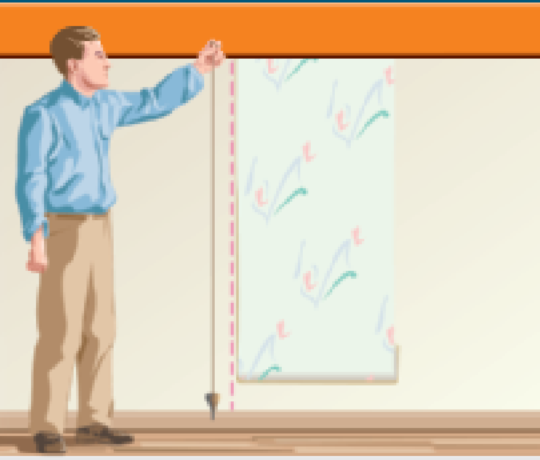
Now

1. Find the distance between a point and a line.
2. Find the distance between parallel lines.

Why?

- A *plumb bob* is made of string with a specially designed weight. When the weight is suspended and allowed to swing freely, the point of the bob is precisely below the point to which the string is fixed.

The plumb bob is useful in establishing what is the true vertical or *plumb* when constructing a wall or when hanging wallpaper.



NewVocabulary
equidistant

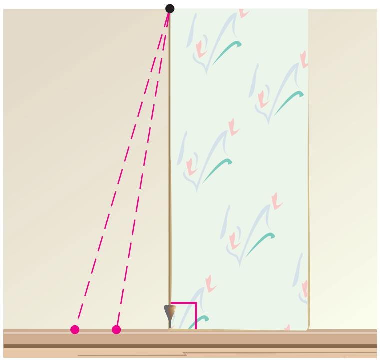
Common Core State Standards

Content Standards
G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).
G.MG.3 Apply geometric methods to solve problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

Mathematical Practices
2 Reason abstractly and quantitatively.
4 Model with mathematics.

1 Distance From a Point to a Line

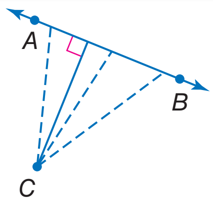
The plumb bob also indicates the shortest distance between the point at which it is attached on the ceiling and a level floor below. This perpendicular distance between a point and a line is the shortest in all cases.



KeyConcept Distance Between a Point and a Line

Words The distance between a line and a point not on the line is the length of the segment perpendicular to the line from the point.

Model

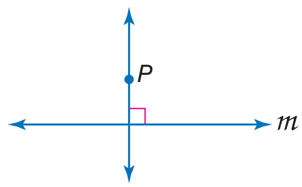


The construction of a line perpendicular to an existing line through a point not on the existing line in Extend Lesson 1-5 establishes that there is *at least one* line through a point P that is perpendicular to a line AB. The following postulate states that this line is the *only* line through P perpendicular to $\overleftrightarrow{AB}$.

Postulate 3.6 Perpendicular Postulate

Words If given a line and a point not on the line, then there exists exactly one line through the point that is perpendicular to the given line.

Model



Real-WorldCareer

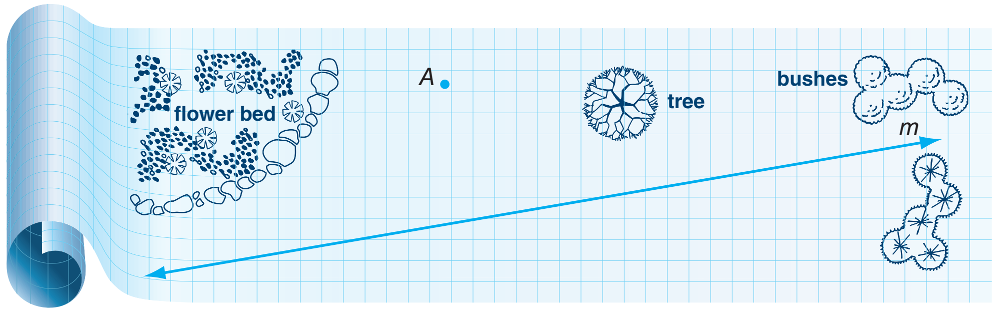
Landscape Architect Landscape architects enjoy working with their hands and possess strong analytical skills. Creative vision and artistic talent are also desirable qualities. Typically, a bachelor's degree is required of landscape architects, but a master's degree may be required for specializations such as golf course design.

StudyTip

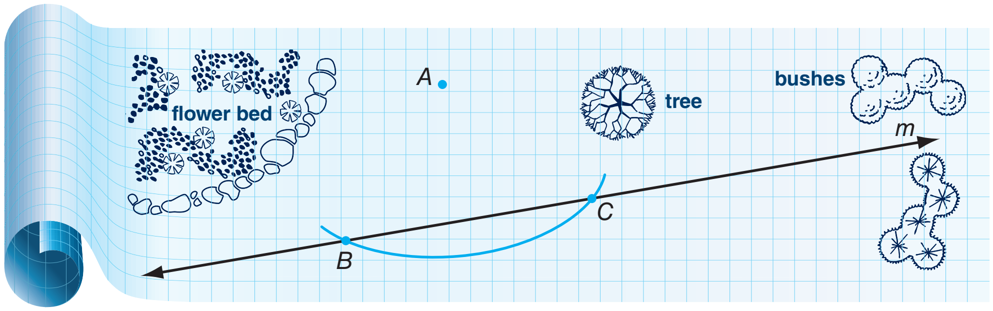
Drawing the Shortest Distance You can use tools like the corner of a piece of paper to help you draw a perpendicular segment from a point to a line, but only a compass and a straightedge can be used to construct this segment.

Real-World Example 1 Construct Distance From a Point to a Line

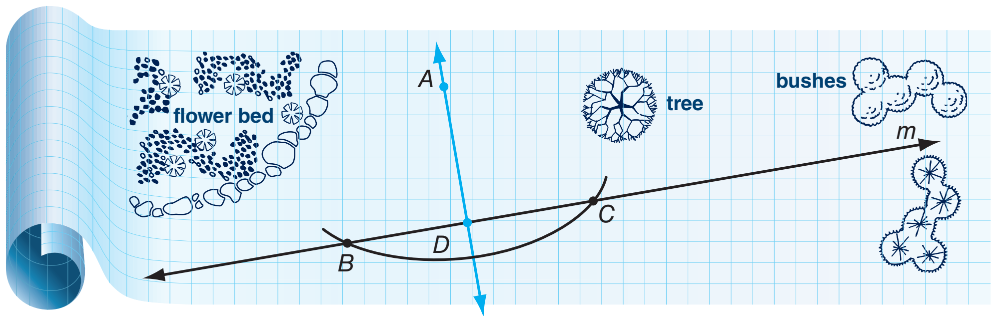
LANDSCAPING A landscape architect notices that one part of a yard does not drain well. She wants to tap into an existing underground drain represented by line *m*. Construct and name the segment with the length that represents the shortest amount of pipe she will need to lay to connect this drain to point *A*.



The distance from a line to a point not on the line is the length of the segment perpendicular to the line from the point. Locate points B and C on line m equidistant from point A.



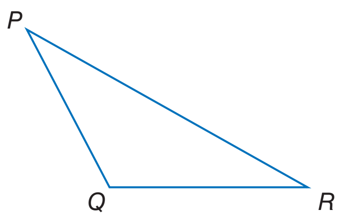
Locate a third point on line m equidistant from B and C. Label this point D. Then draw $\overleftrightarrow{AD}$ so that $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.



The measure of $\overline{AD}$ represents the shortest amount of pipe the architect will need to lay to connect the drain to point A.

GuidedPractice

1. Copy the figure. Then construct and name the segment that represents the distance from Q to $\overleftrightarrow{PR}$.



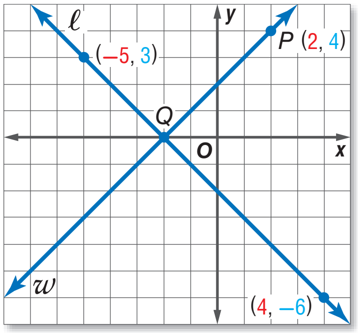
Ocean/CORBIS

Study Tip

Distance to Axes Note that the distance from a point to the x-axis can be determined by looking at the y-coordinate, and the distance from a point to the y-axis can be determined by looking at the x-coordinate.

Example 2 Distance from a Point to a Line on Coordinate Plane

COORDINATE GEOMETRY **Line ℓ contains points at $(-5, 3)$ and $(4, -6)$. Find the distance between line ℓ and point $P(2, 4)$.**



Step 1 Find the equation of the line ℓ.

Begin by finding the slope of the line through points $(-5, 3)$ and $(4, -6)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6 - 3}{4 - (-5)} = \frac{-9}{9} \text{ or } -1$$

Then write the equation of this line using the point $(4, -6)$ on the line.

$y = mx + b$	Slope-intercept form
$-6 = -1(4) + b$	$m = -1, (x, y) = (4, -6)$
$-6 = -4 + b$	Simplify.
$-2 = b$	Add 4 to each side.

The equation of line ℓ is $y = -x + (-2)$ or $y = -x - 2$.

Step 2 Write an equation of the line w perpendicular to line ℓ through $P(2, 4)$.

Since the slope of line ℓ is -1, the slope of a line p is 1. Write the equation of line w through $P(2, 4)$ with slope 1.

$y = mx + b$	Slope-intercept form
$4 = 1(2) + b$	$m = -1, (x, y) = (2, 4)$
$4 = 2 + b$	Simplify.
$2 = b$	Subtract 2 from each side.

The equation of line w is $y = x + 2$.

Study Tip

Elimination Method To review solving systems of equations using the elimination method, see p. P18.

Step 3 Solve the system of equations to determine the point of intersection.

$$\begin{array}{lrl} \text{line } \ell: & y &= -x - 2 \\ \text{line } w: & \underline{(+)\ y} &\underline{= x + 2} \\ & 2y &= 0 \\ & y &= 0 \end{array}$$

Add the two equations.

Divide each side by 2.

Solve for x.

$0 = x + 2$	Substitute 0 for y in the second equation.
$-2 = x$	Subtract 2 from each side.

The point of intersection is $(-2, 0)$. Let this be point Q.

Step 4 Use the Distance Formula to determine the distance between $P(2, 4)$ and $Q(-2, 0)$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance formula
$= \sqrt{(-2 - 2)^2 + (0 - 4)^2}$	$x_2 = -2, x_1 = 2, y_2 = 0, y_1 = 4$
$= \sqrt{32}$	Simplify.

The distance between the point and the line is $\sqrt{32}$ or about 5.66 units.

GuidedPractice

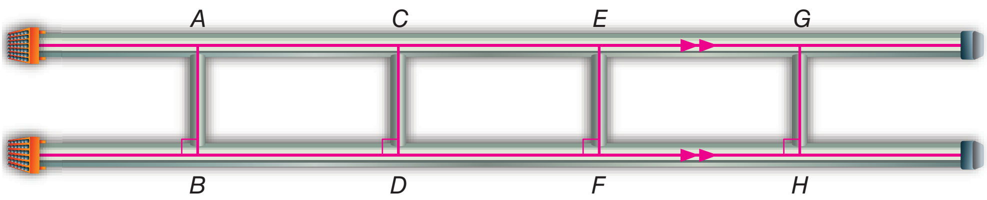
2. Line ℓ contains points at (1, 2) and (5, 4). Construct a line perpendicular to ℓ through $P(1, 7)$. Then find the distance from P to ℓ.

StudyTip

Equidistant You will use this concept of *equidistant* to describe special points and lines relating to the sides and angles of triangles in Lesson 5-1.

2 Distance Between Parallel Lines

By definition, parallel lines do not intersect. An alternate definition states that two lines in a plane are parallel if they are everywhere **equidistant**. Equidistant means that the distance between two lines measured along a perpendicular line to the lines is always the same.



$AB = CD = EF = GH$

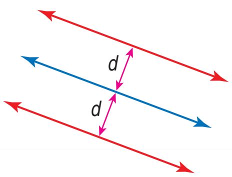
This leads to the definition of the distance between two parallel lines.

KeyConcept Distance Between Parallel Lines

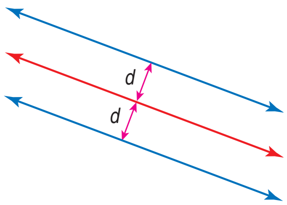
The distance between two parallel lines is the perpendicular distance between one of the lines and any point on the other line.

StudyTip

Locus of Points Equidistant from Two Parallel Lines Conversely, the locus of points in a plane that are equidistant from two parallel lines is a third line that is parallel to and centered between the two parallel lines.



Recall from Lesson 1-1 that a *locus* is the set of all points that satisfy a given condition. Parallel lines can be described as the locus of points in a plane equidistant from a given line.



Theorem 3.9 Two Lines Equidistant from a Third

In a plane, if two lines are each equidistant from a third line, then the two lines are parallel to each other.

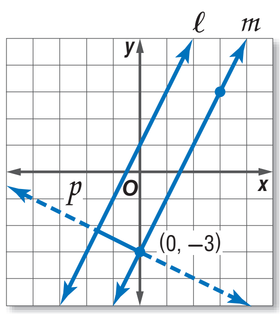
You will prove Theorem 3.9 in Exercise 30.

Example 3 Distance Between Parallel Lines

Find the distance between the parallel lines ℓ and m with equations $y = 2x + 1$ and $y = 2x - 3$, respectively.

You will need to solve a system of equations to find the endpoints of a segment that is perpendicular to both ℓ and m. From their equations, we know that the slope of line ℓ and line m is 2.

Sketch line p through the y-intercept of line m, $(0, -3)$, perpendicular to lines m and ℓ.



Step 1 Write an equation of line p. The slope of p is the opposite reciprocal of 2, or $-\frac{1}{2}$. Use the y-intercept of line m, $(0, -3)$, as one of the endpoints of the perpendicular segment.

$(y - y_1) = m(x - x_1)$	Point-slope form
$[y - (-3)] = -\frac{1}{2}(x - 0)$	$x_1 = 0$, $y_1 = 3$, and $m = -\frac{1}{2}$
$y + 3 = -\frac{1}{2}x$	Simplify.
$y = -\frac{1}{2}x - 3$	Subtract 3 from each side.

Step 2 Use a system of equations to determine the point of intersection of lines ℓ and p.

ℓ: $y = 2x + 1$

p: $y = -\frac{1}{2}x - 3$

$2x + 1 = -\frac{1}{2}x - 3$	Substitute $2x + 1$ for y in the second equation.
$2x + \frac{1}{2}x = -3 - 1$	Group like terms on each side.
$\frac{5}{2}x = -4$	Simplify on each side.
$x = -\frac{8}{5}$	Multiply each side by $\frac{2}{5}$.
$y = -\frac{1}{2}\left(-\frac{8}{5}\right) - 3$	Substitute $-\frac{8}{5}$ for x in the equation for p.
$= -\frac{11}{5}$	Simplify.

The point of intersection is $\left(-\frac{8}{5}, -\frac{11}{5}\right)$ or $(-1.6, -2.2)$.

Step 3 Use the Distance Formula to determine the distance between $(0, -3)$ and $(-1.6, -2.2)$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{(-1.6 - 0)^2 + [-2.2 - (-3)]^2}$	$x_2 = -1.6$, $x_1 = 0$, $y_2 = -2.2$, and $y_1 = -3$
≈ 1.8	Simplify using a calculator.

The distance between the lines is about 1.8 units.

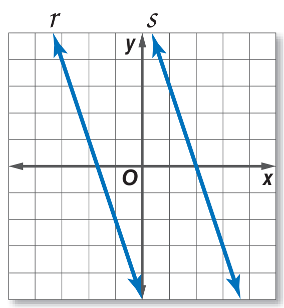
StudyTip

Substitution Method To review **solving systems of equations** using the substitution method, see p. P17.

GuidedPractice

3A. Find the distance between the parallel lines r and s whose equations are $y = -3x - 5$ and $y = -3x + 6$, respectively.

3B. Find the distance between parallel lines a and b with equations $x + 3y = 6$ and $x + 3y = -14$, respectively.

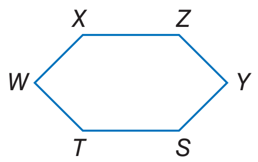


Check Your Understanding

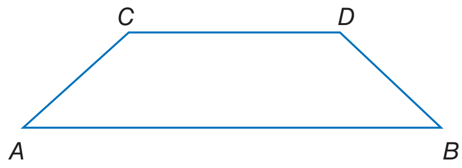
= Step-by-Step Solutions begin on page R14.

Example 1 **Copy each figure. Construct the segment that represents the distance indicated.**

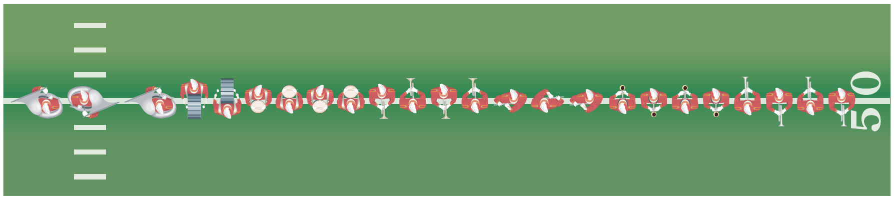
1. Y to $\overleftrightarrow{TS}$



2. C to $\overleftrightarrow{AB}$



3. CCSS **STRUCTURE** After forming a line, every even member of a marching band turns to face the home team's end zone and marches 5 paces straight forward. At the same time, every odd member turns in the opposite direction and marches 5 paces straight forward. Assuming that each band member covers the same distance, what formation should result? Justify your answer.



Example 2 **COORDINATE GEOMETRY** **Find the distance from P to ℓ.**

4. Line ℓ contains points (4, 3) and (−2, 0). Point P has coordinates (3, 10).

5. Line ℓ contains points (−6, 1) and (9, −4). Point P has coordinates (4, 1).

6. Line ℓ contains points (4, 18) and (−2, 9). Point P has coordinates (−9, 5).

Example 3 **Find the distance between each pair of parallel lines with the given equations.**

7. $y = -2x + 4$
$y = -2x + 14$

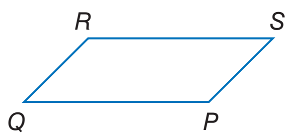
8. $y = 7$
$y = -3$

Practice and Problem Solving

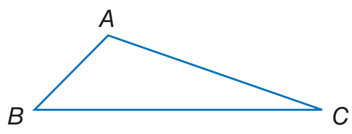
Extra Practice is on page R3.

Example 1 **Copy each figure. Construct the segment that represents the distance indicated.**

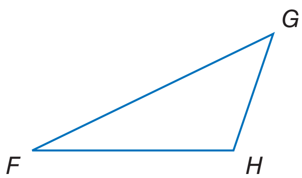
9. Q to $\overline{RS}$



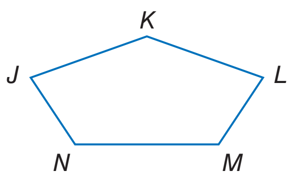
10. A to $\overline{BC}$



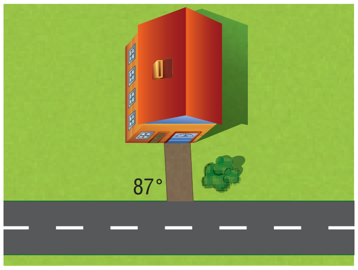
11. H to $\overline{FG}$



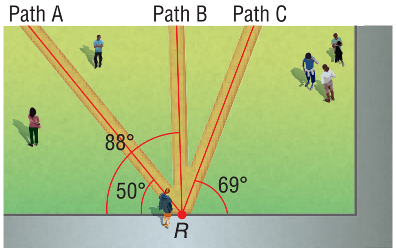
12. K to $\overline{LM}$



13. **DRIVEWAYS** In the diagram at the right, is the driveway shown the shortest possible one from the house to the road? Explain why or why not.



14. **CCSS MODELING** Rondell is crossing the courtyard in front of his school. Three possible paths are shown in the diagram at the right. Which of the three paths shown is the shortest? Explain your reasoning.



Example 2 **COORDINATE GEOMETRY** **Find the distance from P to ℓ.**

15. Line ℓ contains points (0, −3) and (7, 4). Point P has coordinates (4, 3).

16. Line ℓ contains points (11, −1) and (−3, −11). Point P has coordinates (−1, 1).

17. Line ℓ contains points (−2, 1) and (4, 1). Point P has coordinates (5, 7).

18. Line ℓ contains points (4, −1) and (4, 9). Point P has coordinates (1, 6).

19. Line ℓ contains points (1, 5) and (4, −4). Point P has coordinates (−1, 1).

20. Line ℓ contains points (−8, 1) and (3, 1). Point P has coordinates (−2, 4).

Example 3 **Find the distance between each pair of parallel lines with the given equations.**

21. $y = -2$
$y = 4$

22. $x = 3$
$x = 7$

23. $y = 5x - 22$
$y = 5x + 4$

24. $y = \frac{1}{3}x - 3$
$y = \frac{1}{3}x + 2$

25. $x = 8.5$
$x = -12.5$

26. $y = 15$
$y = -4$

27. $y = \frac{1}{4}x + 2$
$4y - x = -60$

28. $3x + y = 3$
$y + 17 = -3x$

29. $y = -\frac{5}{4}x + 3.5$
$4y + 10.6 = -5x$

30. **PROOF** Write a two-column proof of Theorem 3.9.

Find the distance from the line to the given point.

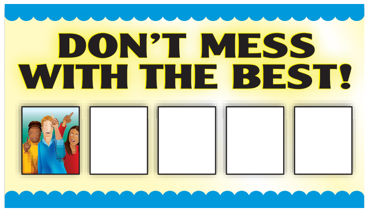
31. $y = -3$, (5, 2)

32. $y = \frac{1}{6}x + 6$, (−6, 5)

33. $x = 4$, (−2, 5)

34. **POSTERS** Alma is hanging two posters on the wall in her room as shown. How can Alma use perpendicular distances to confirm that the posters are parallel?

35. **SCHOOL SPIRIT** Brock is decorating a hallway bulletin board to display pictures of students demonstrating school spirit. He cuts off one length of border to match the width of the top of the board, and then uses that strip as a template to cut a second strip that is exactly the same length for the bottom.

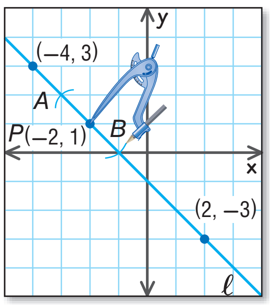


When stapling the bottom border in place, he notices that the strip he cut is about a quarter of an inch too short. Describe what he can conclude about the bulletin board. Explain your reasoning.

CONSTRUCTION **Line ℓ contains points at $(-4, 3)$ and $(2, -3)$. Point P at $(-2, 1)$ is on line ℓ. Complete the following construction.**

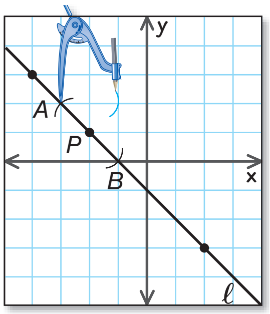
Step 1

Graph line ℓ and point P, and put the compass at point P. Using the same compass setting, draw arcs to the left and right of P. Label these points A and B.



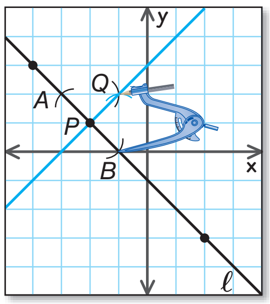
Step 2

Open the compass to a setting greater than AP. Put the compass at point A and draw an arc above line ℓ.



Step 3

Using the same compass setting, put the compass at point B and draw an arc above line ℓ. Label the point of intersection Q. Then draw $\overleftrightarrow{PQ}$.



36. What is the relationship between line ℓ and $\overleftrightarrow{PQ}$? Verify your conjecture using the slopes of the two lines.

37. Repeat the construction above using a different line and point on that line.

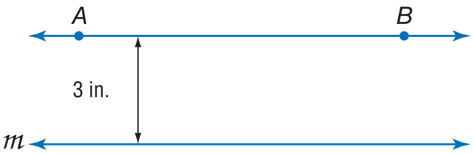
38. CCSS **SENSE-MAKING** $\overline{AB}$ has a slope of 2 and midpoint $M(3, 2)$. A segment perpendicular to $\overline{AB}$ has midpoint $P(4, -1)$ and shares endpoint B with $\overline{AB}$.

a. Graph the segments.

b. Find the coordinates of A and B.

39. **MULTIPLE REPRESENTATIONS** In this problem, you will explore the areas of triangles formed by points on parallel lines.

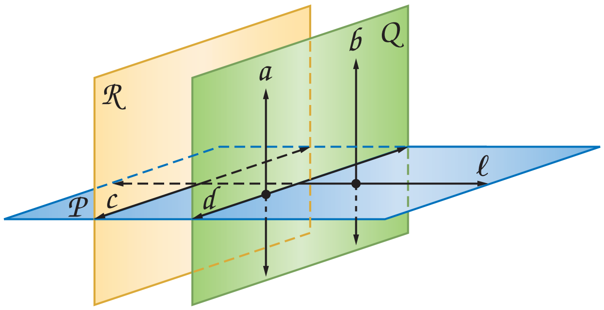
a. Geometric Draw two parallel lines and label them as shown.



b. Verbal Where would you place point C on line m to ensure that triangle ABC would have the largest area? Explain your reasoning.

c. Analytical If $AB = 11$ inches, what is the maximum area of $\triangle ABC$?

40. **PERPENDICULARITY AND PLANES** Make a copy of the diagram below to answer each question, marking the diagram with the given information.



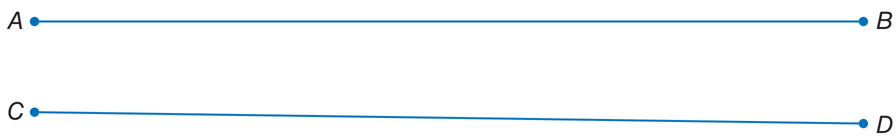
a. If two lines are perpendicular to the same plane, then they are coplanar. If both line a and line b are perpendicular to plane $\mathcal{P}$, what must also be true?

b. If a plane intersects two parallel planes, then the intersections form two parallel lines. If planes $\mathcal{R}$ and Q are parallel and they intersect plane $\mathcal{P}$, what must also be true?

c. If two planes are perpendicular to the same line, then they are parallel. If both plane Q and plane $\mathcal{R}$ are perpendicular to line ℓ, what must also be true?

H.O.T. Problems Use Higher-Order Thinking Skills

41. **ERROR ANALYSIS** Han draws the segments $\overline{AB}$ and $\overline{CD}$ shown below using a straightedge. He claims that these two lines, if extended, will never intersect. Shenequa claims that they will. Is either of them correct? Justify your answer.



42. **CHALLENGE** Describe the locus of points that are equidistant from two intersecting lines, and sketch an example.

43. **CHALLENGE** Suppose a line perpendicular to a pair of parallel lines intersects the lines at the points $(a, 4)$ and $(0, 6)$. If the distance between the parallel lines is $\sqrt{5}$, find the value of a and the equations of the parallel lines.

44. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain.

The distance between a line and a plane can be found.

45. **OPEN ENDED** Draw an irregular convex pentagon using a straightedge.

a. Use a compass and straightedge to construct a line between one vertex and a side opposite the vertex.

b. Use measurement to justify that the line constructed is perpendicular to the side chosen.

c. Use mathematics to justify this conclusion.

46. CCSS **SENSE-MAKING** Rewrite Theorem 3.9 in terms of two planes that are equidistant from a third plane. Sketch an example.

47. **WRITING IN MATH** Summarize the steps necessary to find the distance between a pair of parallel lines given the equations of the two lines.

Standardized Test Practice

48. EXTENDED RESPONSE Segment AB is perpendicular to segment CD. Segment AB and segment CD bisect each other at point X.

a. Draw a figure to represent the problem.

b. Find $\overline{BD}$ if $AB = 12$ and $CD = 16$.

c. Find $\overline{BD}$ if $AB = 24$ and $CD = 18$.

49. A city park is square and has an area of 81,000 square feet. Which of the following is the closest to the length of one side of the park?

A 100 ft **C** 300 ft
B 200 ft **D** 400 ft

50. ALGEBRA Pablo bought a sweater on sale for 25% off the original price and another 40% off the discounted price. If the sweater originally cost \$48, what was the final price of the sweater?

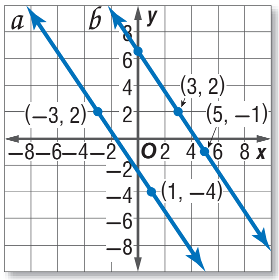
F \$14.40 **H** \$31.20
G \$21.60 **J** \$36.00

51. SAT/ACT After N cookies are divided equally among 8 children, 3 remain. How many would remain if $(N + 6)$ cookies were divided equally among the 8 children?

A 0 **C** 2 **E** 6
B 1 **D** 4

Spiral Review

52. Refer to the figure at the right. Determine whether $a \parallel b$. Justify your answer. (Lesson 3-5)



Write an equation in point-slope form of the line having the given slope that contains the given point. (Lesson 3-4)

53. m: $\frac{1}{4}$, $(3, -1)$

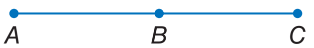
54. m: 0, $(-2, 6)$

55. m: -1, $(-2, 3)$

56. m: -2, $(-6, -7)$

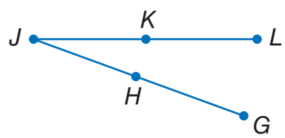
Prove the following. (Lesson 2-7)

57. If $AB = BC$, then $AC = 2BC$.

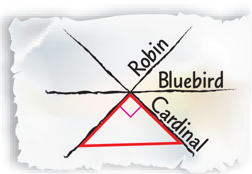


58. Given: $\overline{JK} \cong \overline{KL}$, $\overline{HJ} \cong \overline{GH}$, $\overline{KL} \cong \overline{HJ}$

Prove: $\overline{GH} \cong \overline{JK}$



59. MAPS Darnell sketched a map for his friend of the cross streets nearest to his home. Describe two different angle relationships between the streets. (Lesson 1-5)



Skills Review

Use the Distance Formula to find the distance between each pair of points.

60. $A(0, 0)$, $B(15, 20)$

61. $O(-12, 0)$, $P(-8, 3)$

62. $C(11, -12)$, $D(6, 2)$

63. $R(-2, 3)$, $S(3, 15)$

64. $M(1, -2)$, $N(9, 13)$

65. $Q(-12, 2)$, $T(-9, 6)$