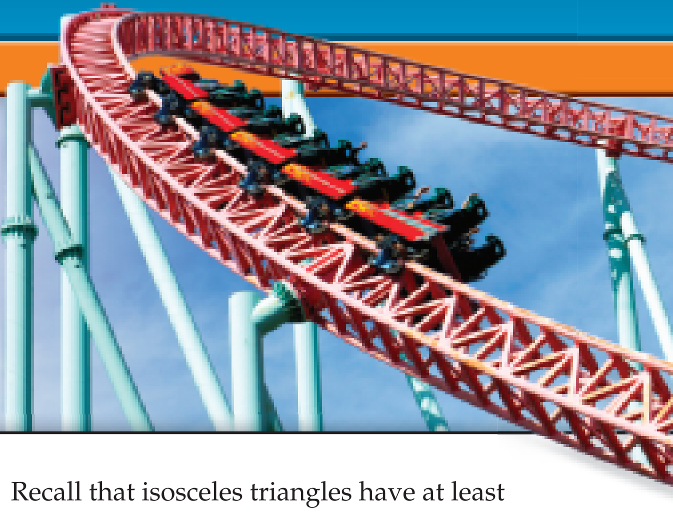
LESSON 4-6 Isosceles and Equilateral Triangles



Then
- You identified isosceles and equilateral triangles.

Now
1. Use properties of isosceles triangles.
2. Use properties of equilateral triangles.

Why?
- The tracks on the roller coaster have triangular reinforcements between the tracks for support and stability. The triangle supports in the photo are isosceles triangles.

NewVocabulary
legs of an isosceles triangle
vertex angle
base angles

Common Core State Standards

Content Standards

G.CO.10 Prove theorems about triangles.

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Mathematical Practices

2 Reason abstractly and quantitatively.

3 Construct viable arguments and critique the reasoning of others.

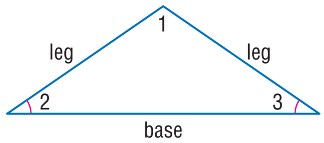
1 Properties of Isosceles Triangles

Recall that isosceles triangles have at least two congruent sides. The parts of an isosceles triangle have special names.

The two congruent sides are called the **legs of an isosceles triangle**, and the angle with sides that are the legs is called the **vertex angle**. The side of the triangle opposite the vertex angle is called the *base*. The two angles formed by the base and the congruent sides are called the **base angles**.

$\angle 1$ is the vertex angle.

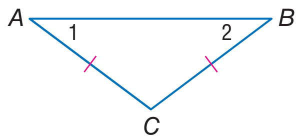
$\angle 2$ and $\angle 3$ are the base angles.



Theorems Isosceles Triangle

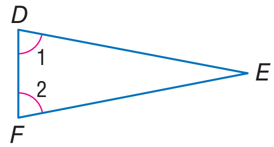
4.10 Isosceles Triangle Theorem
If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

Example If $\overline{AC} \cong \overline{BC}$, then $\angle 2 \cong \angle 1$.



4.11 Converse of Isosceles Triangle Theorem
If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

Example If $\angle 1 \cong \angle 2$, then $\overline{FE} \cong \overline{DE}$.



You will prove Theorem 4.11 in Exercise 37.

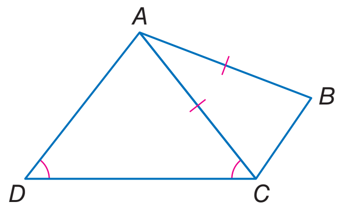
Example 1 Congruent Segments and Angles

a. Name two unmarked congruent angles.

$\angle ACB$ is opposite $\overline{AB}$ and $\angle B$ is opposite $\overline{AC}$, so $\angle ACB \cong \angle B$.

b. Name two unmarked congruent segments.

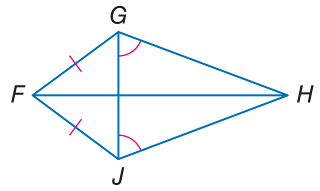
$\overline{AD}$ is opposite $\angle ACD$ and $\overline{AC}$ is opposite $\angle D$, so $\overline{AD} \cong \overline{AC}$.



GuidedPractice

1A. Name two unmarked congruent angles.

1B. Name two unmarked congruent segments.

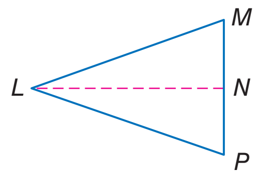


To prove the Isosceles Triangle Theorem, draw an auxiliary line and use the two triangles formed.

Proof Isosceles Triangle Theorem

Given: $\triangle LMP$; $\overline{LM} \cong \overline{LP}$

Prove: $\angle M \cong \angle P$



Proof:

Statements	Reasons
1. Let N be the midpoint of $\overline{MP}$.	**1.** Every segment has exactly one midpoint.
2. Draw an auxiliary segment $\overline{LN}$.	**2.** Two points determine a line.
3. $\overline{MN} \cong \overline{PN}$	**3.** Midpoint Theorem
4. $\overline{LN} \cong \overline{LN}$	**4.** Reflexive Property of Congruence
5. $\overline{LM} \cong \overline{LP}$	**5.** Given
6. $\triangle LMN \cong \triangle LPN$	**6.** SSS
7. $\angle M \cong \angle P$	**7.** CPCTC

2 Properties of Equilateral Triangles

The Isosceles Triangle Theorem leads to two corollaries about the angles of an equilateral triangle.

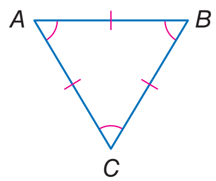
ReviewVocabulary

equilateral triangle a triangle with three congruent sides

Corollaries Equilateral Triangle

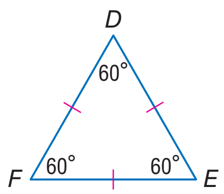
4.3 A triangle is equilateral if and only if it is equiangular.

Example If $\angle A \cong \angle B \cong \angle C$, then $\overline{AB} \cong \overline{BC} \cong \overline{CA}$.



4.4 Each angle of an equilateral triangle measures 60.

Example If $\overline{DE} \cong \overline{EF} \cong \overline{FE}$, then $m\angle A = m\angle B = m\angle C = 60$.

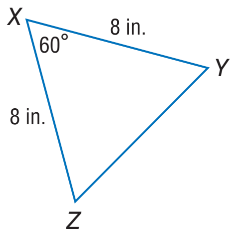


You will prove Corollaries 4.3 and 4.4 in Exercises 35 and 36.

Example 2 Find Missing Measures

Find each measure.

a. $m\angle Y$



Since $XY = XZ$, $\overline{XY} \cong \overline{XZ}$. By the Isosceles Triangle Theorem, base angles Z and Y are congruent, so $m\angle Z = m\angle Y$. Use the Triangle Sum Theorem to write and solve an equation to find $m\angle Y$.

$m\angle X + m\angle Y + m\angle Z = 180$	Triangle Sum Theorem
$60 + m\angle Y + m\angle Y = 180$	$m\angle X = 60$, $m\angle Z = m\angle Y$
$60 + 2(m\angle Y) = 180$	Simplify.
$2(m\angle Y) = 120$	Subtract 60 from each side.
$m\angle Y = 60$	Divide each side by 2.

b. YZ

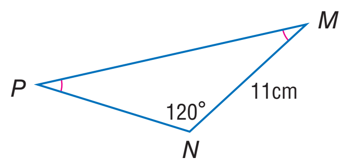
$m\angle Z = m\angle Y$, so $m\angle Z = 60$ by substitution. Since $m\angle X = 60$, all three angles measure 60, so the triangle is equiangular. Because an equiangular triangle is also equilateral, $XY = XZ = ZY$. Since $XY = 8$ inches, $YZ = 8$ inches by substitution.

StudyTip

Isosceles Triangles As you discovered in Example 2, any isosceles triangle that has one 60° angle must be an equilateral triangle.

GuidedPractice

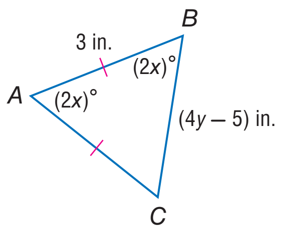
2A. $m\angle M$ **2B.** PN



You can use the properties of equilateral triangles and algebra to find missing values.

Example 3 Find Missing Values

ALGEBRA **Find the value of each variable.**



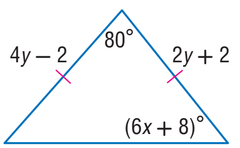
Since $\angle B = \angle A$, $\overline{AC} \cong \overline{BC}$ by the Converse of the Isosceles Triangle Theorem. All of the sides of the triangle are congruent, so the triangle is equilateral. Each angle of an equilateral triangle measures 60°, so $2x = 60$ and $x = 30$.

The triangle is equilateral, so all of the sides are congruent, and the lengths of all of the sides are equal.

$AB = BC$	Definition of equilateral triangle
$3 = 4y - 5$	Substitution
$8 = 4y$	Add 5 to each side.
$2 = y$	Divide each side by 4.

GuidedPractice

3. Find the value of each variable.



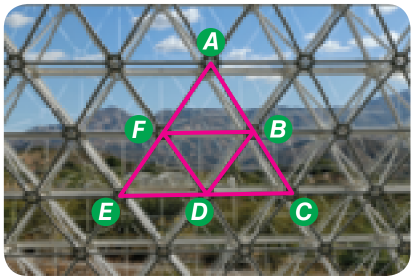
Real-World Link

Biosphere II is the largest totally enclosed ecosystem ever built, covering 3.14 acres in Oracle, Arizona. The controlled-environment facility is 91 feet at its highest point, and it has 6500 windows that enclose a volume of 7.2 million cubic feet.

Source: University of Arizona

Real-World Example 4 Apply Triangle Congruence

ENVIRONMENT **Refer to the photo of Biosphere II at the right. $\triangle ACE$ is an equilateral triangle. F is the midpoint of $\overline{AE}$, D is the midpoint of $\overline{EC}$, and B is the midpoint of $\overline{CA}$. Prove that $\triangle FBD$ is also equilateral.**



Given: $\triangle ACE$ is equilateral. F is the midpoint of $\overline{AE}$, D is the midpoint of $\overline{EC}$, and B is the midpoint of $\overline{CA}$.

Prove: $\triangle FBD$ is equilateral.

Proof:

Statements	Reasons
1. $\triangle ACE$ is equilateral.	**1.** Given
2. F is the midpoint of AE, D is the midpoint of EC, and B is the midpoint of CA.	**2.** Given
3. $m\angle A = 60$, $m\angle C = 60$, $m\angle E = 60$	**3.** Each angle of an equilateral triangle measures 60.
4. $\angle A \cong \angle C \cong \angle E$	**4.** Definition of congruence and substitution
5. $\overline{AE} \cong \overline{EC} \cong \overline{CA}$	**5.** Definition of equilateral triangle
6. $AE = EC = CA$	**6.** Definition of congruence
7. $\overline{AF} \cong \overline{FE}$, $\overline{ED} \cong \overline{DC}$, $\overline{CB} \cong \overline{BA}$	**7.** Midpoint Theorem
8. $AF = FE$, $ED = DC$, $CB = BA$	**8.** Definition of congruence
9. $AF + FE = AE$, $ED + DC = EC$, $CB + BA = CA$	**9.** Segment Addition Postulate
10. $AF + AF = AE$, $FE + FE = AE$, $ED + ED = EC$, $DC + DC = EC$, $CB + CB = CA$, $BA + BA = CA$	**10.** Substitution
11. $2AF = AE$, $2FE = AE$, $2ED = EC$, $2DC = EC$, $2CB = CA$, $2BA = CA$	**11.** Addition Property
12. $2AF = AE$, $2FE = AE$, $2ED = AE$, $2DC = AE$, $2CB = AE$, $2BA = AE$	**12.** Substitution Property
13. $2AF = 2ED = 2CB$, $2FE = 2DC = 2BA$	**13.** Transitive Property
14. $AF = ED = CB$, $FE = DC = BA$	**14.** Division Property
15. $\overline{AF} \cong \overline{ED} \cong \overline{CB}$, $\overline{FE} \cong \overline{DC} \cong \overline{BA}$	**15.** Definition of congruence
16. $\triangle AFB \cong \triangle EDF \cong \triangle CBD$	**16.** SAS
17. $\overline{DF} \cong \overline{FB} \cong \overline{BD}$	**17.** CPCTC
18. $\triangle FBD$ is equilateral.	**18.** Definition of equilateral triangle

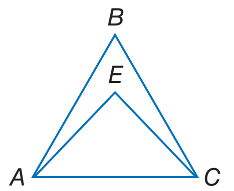
GuidedPractice

4. Given that $\triangle ACE$ is equilateral, $\overline{FB} \parallel \overline{EC}$, $\overline{FD} \parallel \overline{BC}$, $\overline{BD} \parallel \overline{EF}$, and D is the midpoint of $\overline{EC}$, prove that $\triangle FED \cong \triangle BDC$.

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Example 1 **Refer to the figure at the right.**

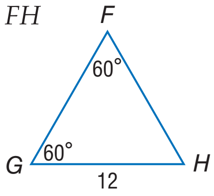


1. If $\overline{AB} \cong \overline{CB}$, name two congruent angles.

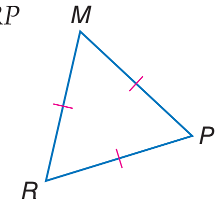
2. If $\angle EAC \cong \angle ECA$, name two congruent segments.

Example 2 **Find each measure.**

3. FH

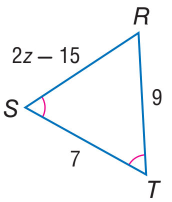


4. $m\angle MRP$



Example 3 CCSS **SENSE-MAKING** **Find the value of each variable.**

5.



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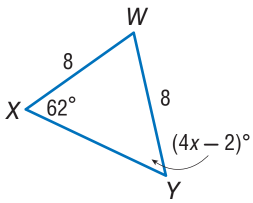
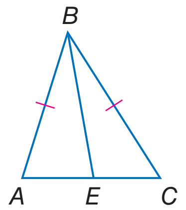


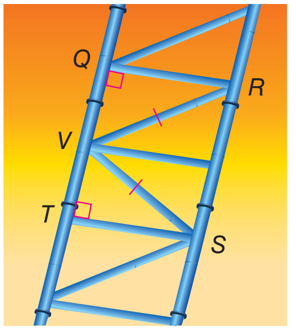
Example 4 **7. PROOF** **Write a two-column proof.**

Given: $\triangle ABC$ is isosceles; $\overline{EB}$ bisects $\angle ABC$.

Prove: $\triangle ABE \cong \triangle CBE$



8. ROLLER COASTERS The roller coaster track shown in the photo on page 285 appears to be composed of congruent triangles. A portion of the track is shown.



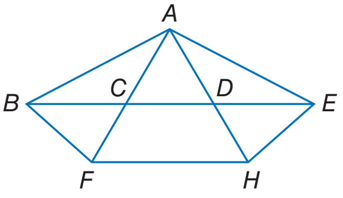
a. If $\overline{QR}$ and $\overline{ST}$ are perpendicular to $\overline{QT}$, $\triangle VSR$ is isosceles with base $\overline{SR}$, and $\overline{QT} \parallel \overline{SR}$, prove that $\triangle RQV \cong \triangle STV$.

b. If $VR = 2.5$ meters and $QR = 2$ meters, find the distance between $\overline{QR}$ and $\overline{ST}$. Explain your reasoning.

Practice and Problem Solving

Extra Practice is on page R4.

Example 1 **Refer to the figure at the right.**



9. If $\overline{AB} \cong \overline{AE}$, name two congruent angles.

10. If $\angle ABF \cong \angle AFB$, name two congruent segments.

11. If $\overline{CA} \cong \overline{DA}$, name two congruent angles.

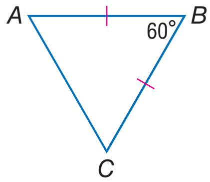
12. If $\angle DAE \cong \angle DEA$, name two congruent segments.

13. If $\angle BCF \cong \angle BFC$, name two congruent segments.

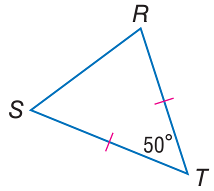
14. If $\overline{FA} \cong \overline{AH}$, name two congruent angles.

Example 2 **Find each measure.**

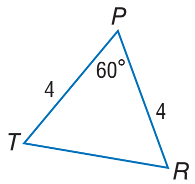
15. $m\angle BAC$



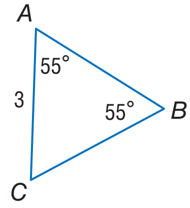
16. $m\angle SRT$



17. TR



18. CB



Example 3 **CCSS REGULARITY** **Find the value of each variable.**

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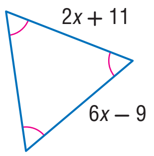

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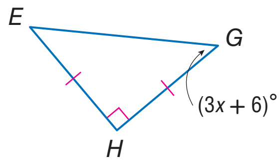

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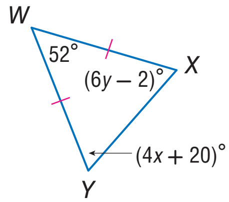

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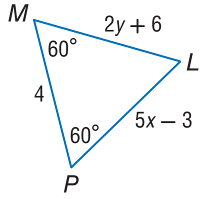
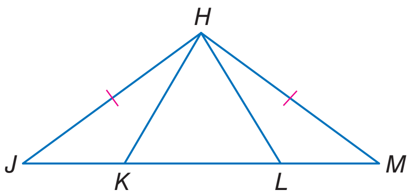


Example 4 **PROOF** **Write a paragraph proof.**

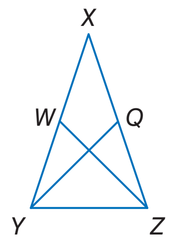
23. Given: $\triangle HJM$ is isosceles, and $\triangle HKL$ is equilateral. $\angle JKH$ and $\angle HKL$ are supplementary and $\angle HLK$ and $\angle MLH$ are supplementary.

Prove: $\angle JHK \cong \angle MHL$

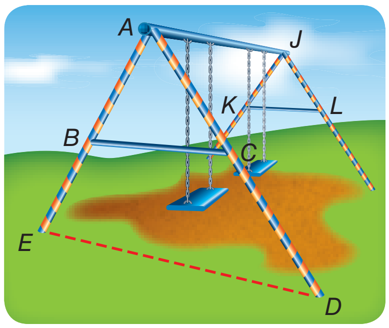


24. Given: $\overline{XY} \cong \overline{XZ}$
W is the midpoint of $\overline{XY}$.
Q is the midpoint of $\overline{XZ}$.

Prove: $\overline{WZ} \cong \overline{QY}$



25. BABYSITTING While babysitting her neighbor's children, Elisa observes that the supports on either side of a park swing set form two sets of triangles. Using a jump rope to measure, Elisa is able to determine that $\overline{AB} \cong \overline{AC}$, but $\overline{BC} \not\cong \overline{AB}$.



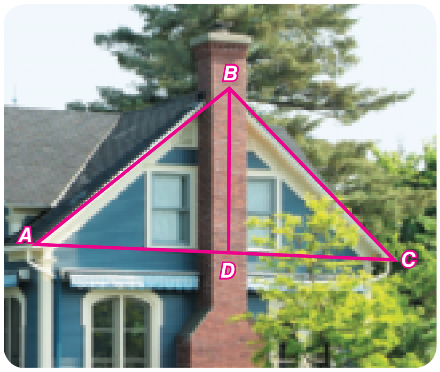
a. Elisa estimates $m\angle BAC$ to be 50. Based on this estimate, what is $m\angle ABC$? Explain.

b. If $\overline{BE} \cong \overline{CD}$, show that $\triangle AED$ is isosceles.

c. If $\overline{BC} \parallel \overline{ED}$ and $\overline{ED} \cong \overline{AD}$, show that $\triangle AED$ is equilateral.

d. If $\triangle JKL$ is isosceles, what is the minimum information needed to prove that $\triangle ABC \cong \triangle JLK$? Explain your reasoning.

26. **CHIMNEYS** In the picture, $\overline{BD} \perp \overline{AC}$ and $\triangle ABC$ is an isosceles triangle with base $\overline{AC}$. Show that the chimney of the house, represented by $\overline{BD}$, bisects the angle formed by the sloped sides of the roof, $\angle ABC$.



27. **CONSTRUCTION** Construct three different isosceles right triangles. Explain your method. Then verify your constructions using measurement and mathematics.

28. **PROOF** Based on your construction in Exercise 27, make and prove a conjecture about the relationship between the base angles of an isosceles right triangle.

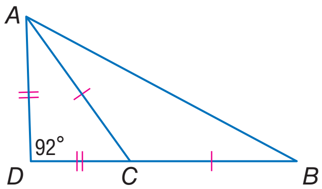
CCSS REGULARITY **Find each measure.**

29. $m\angle CAD$

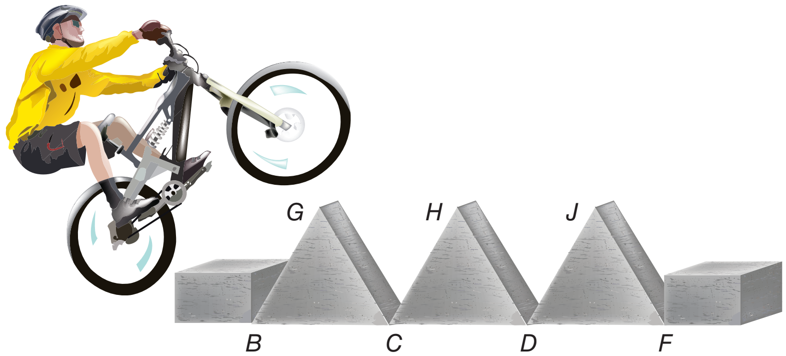
30. $m\angle ACD$

31. $m\angle ACB$

32. $m\angle ABC$

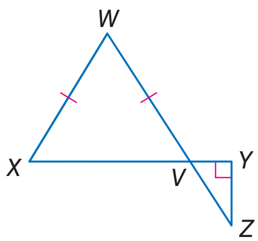


33. **FITNESS** In the diagram, the rider will use his bike to hop across the tops of each of the concrete solids shown. If each triangle is isosceles with vertex angles G, H, and J, and $\overline{BG} \cong \overline{HC}$, $\overline{HD} \cong \overline{JF}$, $\angle G \cong \angle H$, and $\angle H \cong \angle J$, show that the distance from B to F is three times the distance from D to F.



34. **Given:** $\triangle XWV$ is isosceles; $\overline{ZY} \perp \overline{YV}$.

Prove: $\angle X$ and $\angle YZV$ are complementary.



PROOF **Write a two-column proof of each corollary or theorem.**

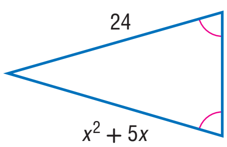
35. Corollary 4.3

36. Corollary 4.4

37. Theorem 4.11

Find the value of each variable.

38.



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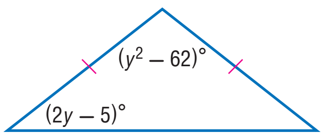


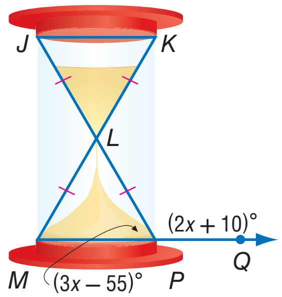
GAMES **Use the diagram of a game timer shown to find each measure.**

40. $m\angle LPM$

41. $m\angle LMP$

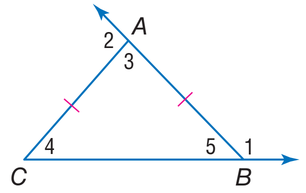
42. $m\angle JLK$

43. $m\angle JKL$



44. **MULTIPLE REPRESENTATIONS** In this problem, you will explore possible measures of the interior angles of an isosceles triangle given the measure of one exterior angle.

a. **Geometric** Use a ruler and a protractor to draw three different isosceles triangles, extending one of the sides adjacent to the vertex angle and to one of the base angles, and labeling as shown.



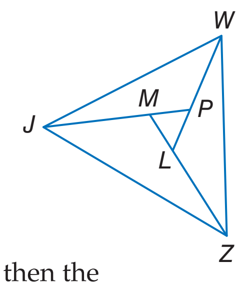
b. **Tabular** Use a protractor to measure and record $m\angle 1$ for each triangle. Use $m\angle 1$ to calculate the measures of $\angle 3$, $\angle 4$, and $\angle 5$. Then find and record $m\angle 2$ and use it to calculate these same measures. Organize your results in two tables.

c. **Verbal** Explain how you used $m\angle 1$ to find the measures of $\angle 3$, $\angle 4$, and $\angle 5$. Then explain how you used $m\angle 2$ to find these same measures.

d. **Algebraic** If $m\angle 1 = x$, write an expression for the measures of $\angle 3$, $\angle 4$, and $\angle 5$. Likewise, if $m\angle 2 = x$, write an expression for these same angle measures.

H.O.T. Problems Use Higher-Order Thinking Skills

45. **CHALLENGE** In the figure at the right, if $\triangle WJZ$ is equilateral and $\angle ZWP \cong \angle WJM \cong \angle JZL$, prove that $\overline{WP} \cong \overline{ZL} \cong \overline{JM}$.

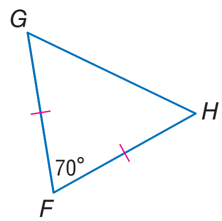


CCSS PRECISION **Determine whether the following statements are *sometimes*, *always*, or *never* true. Explain.**

46. If the measure of the vertex angle of an isosceles triangle is an integer, then the measure of each base angle is an integer.

47. If the measures of the base angles of an isosceles triangle are integers, then the measure of its vertex angle is odd.

48. **ERROR ANALYSIS** Alexis and Miguela are finding $m\angle G$ in the figure shown. Alexis says that $m\angle G = 35$, while Miguela says that $m\angle G = 60$. Is either of them correct? Explain your reasoning.



49. **OPEN ENDED** If possible, draw an isosceles triangle with base angles that are obtuse. If it is not possible, explain why not.

50. **REASONING** In isosceles $\triangle ABC$, $m\angle B = 90$. Draw the triangle. Indicate the congruent sides and label each angle with its measure.

51. **WRITING IN MATH** How can triangle classifications help you prove triangle congruence?

Standardized Test Practice

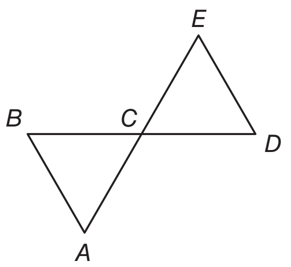
52. ALGEBRA What quantity should be added to both sides of this equation to complete the square?

$$x^2 - 10x = 3$$

A -25 **C** 5
B -5 **D** 25

53. SHORT RESPONSE In a school of 375 students, 150 students play sports and 70 students are involved in the community service club. 30 students play sports and are involved in the community service club. How many students are *not* involved in either sports or the community service club?

54. In the figure $\overline{AE}$ and $\overline{BD}$ bisect each other at point C.



Which additional piece of information would be enough to prove that $\overline{DE} \cong \overline{DC}$?

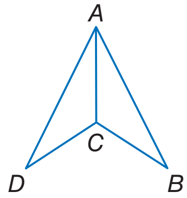
F $\angle A \cong \angle BCA$ **H** $\angle ACB \cong \angle EDC$
G $\angle B \cong \angle D$ **J** $\angle A \cong \angle B$

55. SAT/ACT If $x = -3$, then $4x^2 - 7x + 5 =$

A 2 **C** 20 **E** 62
B 14 **D** 42

Spiral Review

56. If $m\angle ADC = 35$, $m\angle ABC = 35$, $m\angle DAC = 26$, and $m\angle BAC = 26$, determine whether $\triangle ADC \cong \triangle ABC$. (Lesson 4-5)

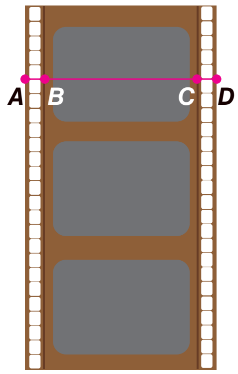


Determine whether $\triangle STU \cong \triangle XYZ$. Explain. (Lesson 4-4)

57. $S(0, 5)$, $T(0, 0)$, $U(1, 1)$, $X(4, 8)$, $Y(4, 3)$, $Z(6, 3)$

58. $S(2, 2)$, $T(4, 6)$, $U(3, 1)$, $X(-2, -2)$, $Y(-4, 6)$, $Z(-3, 1)$

59. PHOTOGRAPHY Film is fed through a traditional camera by gears that catch the perforation in the film. The distance from A to C is the same as the distance from B to D. Show that the two perforated strips are the same width. (Lesson 2-7)



State the property that justifies each statement. (Lesson 2-6)

60. If $x(y + z) = a$, then $xy + xz = a$.

61. If $n - 17 = 39$, then $n = 56$.

62. If $m\angle P + m\angle Q = 110$ and $m\angle R = 110$, then $m\angle P + m\angle Q = m\angle R$.

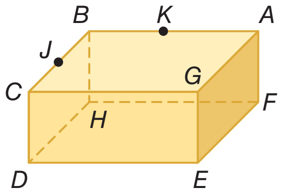
63. If $cv = md$ and $md = 15$, then $cv = 15$.

Refer to the figure at the right. (Lesson 1-1)

64. How many planes appear in this figure?

65. Name three points that are collinear.

66. Are points A, C, D, and J coplanar?



Skills Review

67. PROOF If $\angle ACB \cong \angle ABC$, then $\angle XCA \cong \angle YBA$.