LESSON 5-1 Bisectors of Triangles

Then

- You used segment and angle bisectors.

Now

- 1 Identify and use perpendicular bisectors in triangles.
- 2 Identify and use angle bisectors in triangles.

Why?

- Creating a work triangle in a kitchen can make food preparation more efficient by cutting down on the number of steps you have to take. To locate the point that is equidistant from the sink, stove, and refrigerator, you can use the perpendicular bisectors of the triangle.

NewVocabulary

perpendicular bisector
concurrent lines
point of concurrency
circumcenter
incenter

Common Core State Standards

Content Standards

G.CO.10 Prove theorems about triangles.

G.MG.3 Apply geometric methods to solve problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

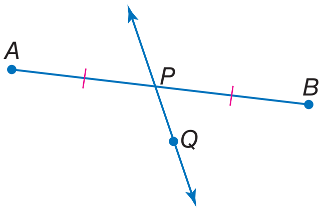
Mathematical Practices

1 Make sense of problems and persevere in solving them.

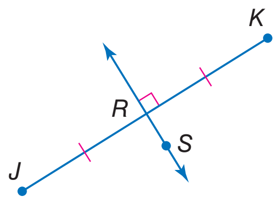
3 Construct viable arguments and critique the reasoning of others.

1 Perpendicular Bisectors

In Lesson 1-3, you learned that a segment bisector is any segment, line, or plane that intersects a segment at its midpoint. If a bisector is also perpendicular to the segment, it is called a **perpendicular bisector**.



$\overleftrightarrow{PQ}$ is a bisector of $\overline{AB}$.



$\overleftrightarrow{RS}$ is a perpendicular bisector of $\overline{JK}$.

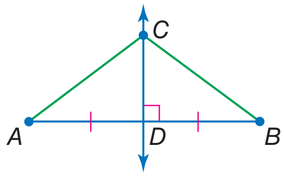
Recall that a *locus* is a set of points that satisfies a particular condition. The perpendicular bisector of a segment is the locus of points in a plane equidistant from the endpoints of the segment. This leads to the following theorems.

Theorems Perpendicular Bisectors

5.1 Perpendicular Bisector Theorem

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

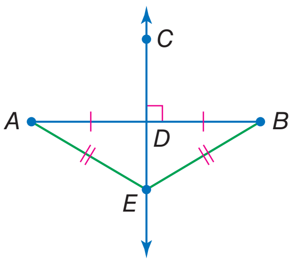
Example: If $\overline{CD}$ is a $\perp$ bisector of $\overline{AB}$, then $AC = BC$.



5.2 Converse of the Perpendicular Bisector Theorem

If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

Example: If $AE = BE$, then E lies on $\overline{CD}$, the $\perp$ bisector of $\overline{AB}$.

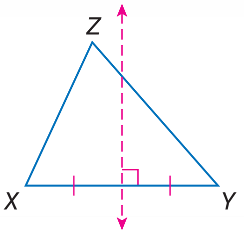


You will prove Theorems 5.1 and 5.2 in Exercises 39 and 37, respectively.

Corbis InsideOutPix/Photolibrary

StudyTip

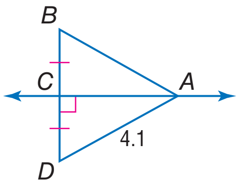
Perpendicular Bisectors The perpendicular bisector of a side of a triangle does not necessarily pass through a vertex of the triangle. For example, in $\triangle XYZ$ below, the perpendicular bisector of $\overline{XY}$ does not pass through point Z.



Example 1 Use the Perpendicular Bisector Theorems

Find each measure.

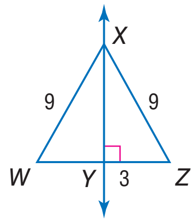
a. AB



From the information in the diagram, we know that $\overleftrightarrow{CA}$ is the perpendicular bisector of $\overline{BD}$.

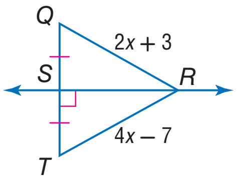
$AB = AD$	Perpendicular Bisector Theorem
$AB = 4.1$	Substitution

b. WY



Since $WX = ZX$ and $\overleftrightarrow{XY} \perp \overline{WZ}$, $\overleftrightarrow{XY}$ is the perpendicular bisector of $\overline{WZ}$ by the Converse of the Perpendicular Bisector Theorem. By the definition of segment bisector, $WY = YZ$. Since $YZ = 3$, $WY = 3$.

c. RT

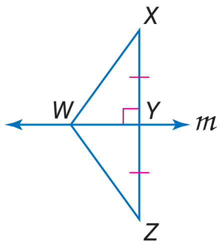


$\overleftrightarrow{SR}$ is the perpendicular bisector of $\overline{QT}$.

$RT = RQ$	Perpendicular Bisector Theorem
$4x - 7 = 2x + 3$	Substitution
$2x - 7 = 3$	Subtract $2x$ from each side.
$2x = 10$	Add 7 to each side.
$x = 5$	Divide each side by 2.

So $RT = 4(5) - 7$ or 13.

GuidedPractice

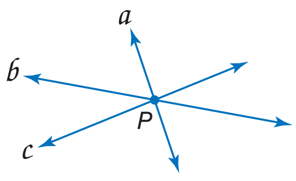


1A. If $WX = 25.3$, $YZ = 22.4$, and $WZ = 25.3$, find XY.

1B. If m is the perpendicular bisector of XZ and $WZ = 14.9$, find WX.

1C. If m is the perpendicular bisector of XZ, $WX = 4a - 15$, and $WZ = a + 12$, find WX.

When three or more lines intersect at a common point, the lines are called **concurrent lines**. The point where concurrent lines intersect is called the **point of concurrency**.



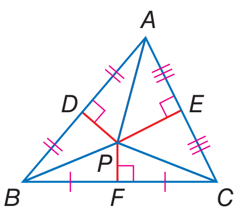
Lines a, b, and c are concurrent at P.

A triangle has three sides, so it also has three perpendicular bisectors. These bisectors are concurrent lines. The point of concurrency of the perpendicular bisectors is called the **circumcenter** of the triangle.

Theorem 5.3 Circumcenter Theorem

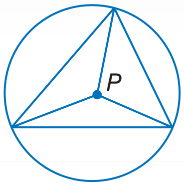
Words The perpendicular bisectors of a triangle intersect at a point called the *circumcenter* that is equidistant from the vertices of the triangle.

Example If P is the circumcenter of $\triangle ABC$, then $PB = PA = PC$.

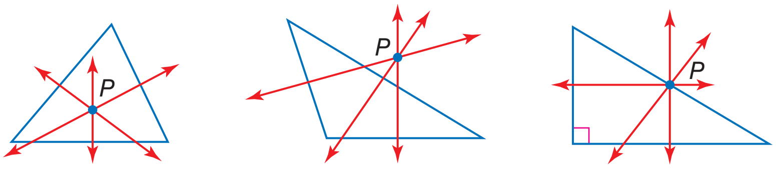


ReadingMath

Circum– The prefix *circum–* means *about* or *around*. The circumcenter is the center of a circle around a triangle that contains the vertices of the triangle.



The circumcenter can be on the interior, exterior, or side of a triangle.

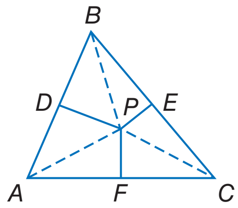


acute triangle   obtuse triangle   right triangle

Proof Circumcenter Theorem

Given: $\overline{PD}$, $\overline{PF}$, and $\overline{PE}$ are perpendicular bisectors of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, respectively.

Prove: $AP = CP = BP$



Paragraph Proof:

Since P lies on the perpendicular bisector of $\overline{AC}$, it is equidistant from A and C. By the definition of equidistant, $AP = CP$. The perpendicular bisector of $\overline{BC}$ also contains P. Thus, $CP = BP$. By the Transitive Property of Equality, $AP = BP$. Thus, $AP = CP = BP$.

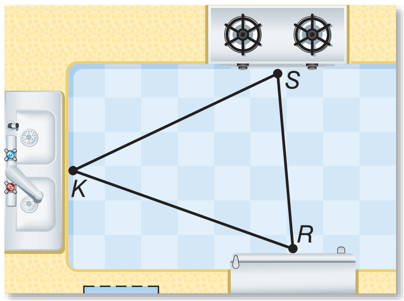
Real-WorldLink

Some basic rules of thumb for a kitchen work triangle are that the sides of the triangle should be no greater than 9 feet and no less than 4 feet. Also, the perimeter of the triangle should be no more than 26 feet and no less than 12 feet.

Source: Merillat

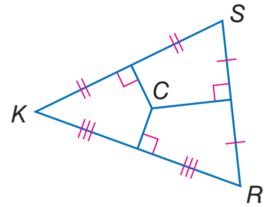
Real-World Example 2 Use the Circumcenter Theorem

INTERIOR DESIGN A stove *S*, sink *K*, and refrigerator *R* are positioned in a kitchen as shown. Find the location for the center of an island work station so that it is the same distance from these three points.



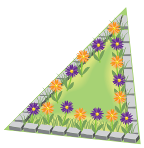
By the Circumcenter Theorem, a point equidistant from three points is found by using the perpendicular bisectors of the triangle formed by those points.

Copy $\triangle SKR$, and use a ruler and protractor to draw the perpendicular bisectors. The location for the center of the island is C, the circumcenter of $\triangle SKR$.



GuidedPractice

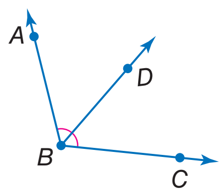
2. To water his triangular garden, Alex needs to place a sprinkler equidistant from each vertex. Where should Alex place the sprinkler?



2 Angle Bisectors

Recall from Lesson 1-4 that an angle bisector divides an angle into two congruent angles. The angle bisector can be a line, segment, or ray.

The bisector of an angle can be described as the locus of points in the interior of the angle equidistant from the sides of the angle. This description leads to the following theorems.



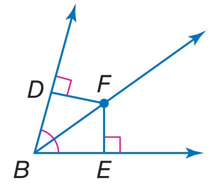
$\overrightarrow{BD}$ is the angle bisector of $\angle ABC$.

Theorems Angle Bisectors

5.4 Angle Bisector Theorem

If a point is on the bisector of an angle, then it is equidistant from the sides of the angle.

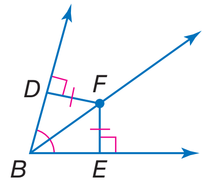
Example: If $\overrightarrow{BF}$ bisects $\angle DBE$, $\overline{FD} \perp \overrightarrow{BD}$, and $\overline{FE} \perp \overrightarrow{BE}$, then $DF = FE$.



5.5 Converse of the Angle Bisector Theorem

If a point in the interior of an angle is equidistant from the sides of the angle, then it is on the bisector of the angle.

Example: If $\overline{FD} \perp \overrightarrow{BD}$, $\overline{FE} \perp \overrightarrow{BE}$, and $DF = FE$, then $\overrightarrow{BF}$ bisects $\angle DBE$.



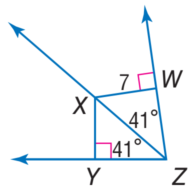
You will prove Theorems 5.4 and 5.5 in Exercises 43 and 40.

Example 3 Use the Angle Bisector Theorems

Find each measure.

a. XY

$XY = XW$	Angle Bisector Theorem
$XY = 7$	Substitution

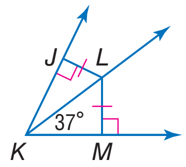


StudyTip

Angle Bisector For part b, only that $JL = LM$ would not be enough information to conclude that $\overrightarrow{KL}$ bisects $\angle JKM$.

b. $m\angle JKL$

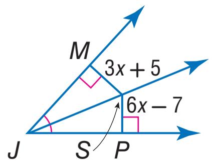
Since $\overline{LJ} \perp \overrightarrow{KJ}$, $\overline{LM} \perp \overrightarrow{KM}$, $\overline{LJ} \cong \overline{LM}$, L is equidistant from the sides of $\angle JKM$. By the Converse of the Angle Bisector Theorem, $\overrightarrow{KL}$ bisects $\angle JKM$.



$\angle JKL \cong \angle LKM$	Definition of angle bisector
$m\angle JKL = m\angle LKM$	Definition of congruent angles
$m\angle JKL = 37$	Substitution

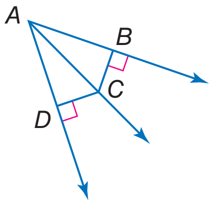
c. SP

$SP = SM$	Angle Bisector Theorem
$6x - 7 = 3x + 5$	Substitution
$3x - 7 = 5$	Subtract $3x$ from each side.
$3x = 12$	Add 7 to each side.
$x = 4$	Divide each side by 3.



So, $SP = 6(4) - 7$ or 17

Guided Practice



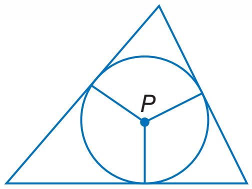
3A. If $m\angle BAC = 38$, $BC = 5$, and $DC = 5$, find $m\angle DAC$.

3B. If $m\angle BAC = 40$, $m\angle DAC = 40$, and $DC = 10$, find BC.

3C. If $\overrightarrow{AC}$ bisects $\angle DAB$, $BC = 4x + 8$, and $DC = 9x - 7$, find BC.

ReadingMath

Incenter The incenter is the center of a circle that intersects each side of the triangle at one point. For this reason, the incenter always lies in the interior of a triangle.

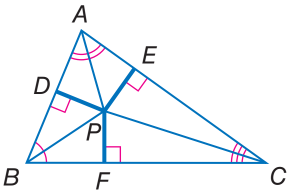


Similar to perpendicular bisectors, since a triangle has three angles, it also has three angle bisectors. The angle bisectors of a triangle are concurrent, and their point of concurrency is called the **incenter** of a triangle.

Theorem 5.6 Incenter Theorem

Words The angle bisectors of a triangle intersect at a point called the *incenter* that is equidistant from the sides of the triangle.

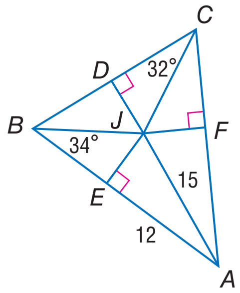
Example If P is the incenter of $\triangle ABC$, then $PD = PE = PF$.



You will prove Theorem 5.6 in Exercise 38.

Example 4 Use the Incenter Theorem

Find each measure if J is the incenter of $\triangle ABC$.



a. JF

By the Incenter Theorem, since J is equidistant from the sides of $\triangle ABC$, $JF = JE$. Find JF by using the Pythagorean Theorem.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$JE^2 + 12^2 = 15^2$	Substitution
$JE^2 + 144 = 225$	$12^2 = 144$ and $15^2 = 225$.
$JE^2 = 81$	Subtract 144 from each side.
$JE = \pm 9$	Take the square root of each side.

Since length cannot be negative, use only the positive square root, 9. Since $JE = JF$, $JF = 9$.

b. $m\angle JAC$

Since $\overrightarrow{BJ}$ bisects $\angle CBE$, $m\angle CBE = 2m\angle JBE$. So $m\angle CBE = 2(34)$ or 68. Likewise, $m\angle DCF = 2m\angle DCJ$, so $m\angle DCF = 2(32)$ or 64.

$m\angle CBE + m\angle DCF + m\angle FAE = 180$	Triangle Angle Sum Theorem
$68 + 64 + m\angle FAE = 180$	$m\angle CBE = 68$, $m\angle DCF = 64$
$132 + m\angle FAE = 180$	Simplify.
$m\angle FAE = 48$	Subtract 132 from each side.

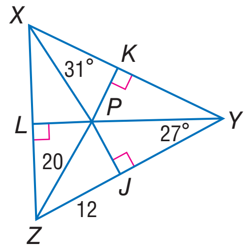
Since $\overrightarrow{AJ}$ bisects $\angle FAE$, $2m\angle JAC = m\angle FAE$. This means that $m\angle JAC = \frac{1}{2}m\angle FAE$, so $m\angle JAC = \frac{1}{2}(48)$ or 24.

GuidedPractice

If P is the incenter of $\triangle XYZ$, find each measure.

4A. PK

4B. $m\angle LZP$

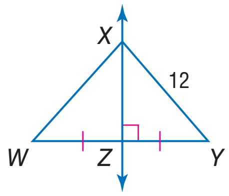


Check Your Understanding

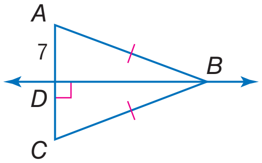
= Step-by-Step Solutions begin on page R14.

Example 1 **Find each measure.**

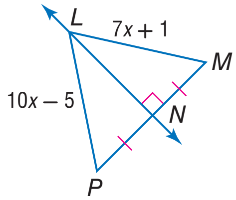
1. *XW*



2. *AC*

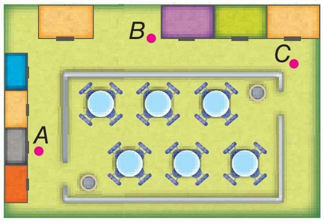


3. *LP*



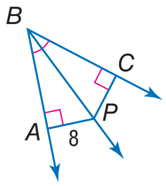
Example 2

4. **ADVERTISING** Four friends are passing out flyers at a mall food court. Three of them take as many flyers as they can and position themselves as shown. The fourth one keeps the supply of additional flyers. Copy the positions of points *A*, *B*, and *C*. Then position the fourth friend at *D* so that she is the same distance from each of the other three friends.

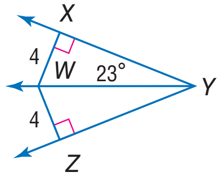


Example 3 **Find each measure.**

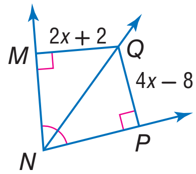
5. *CP*



6. $m\angle WYZ$

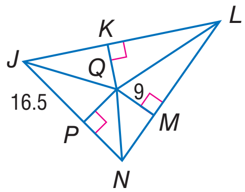


7. *QM*



Example 4

8. **CCSS SENSE-MAKING** Find *JQ* if *Q* is the incenter of $\triangle JLN$.

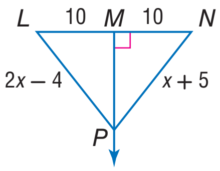


Practice and Problem Solving

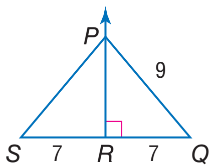
Extra Practice is on page R5.

Example 1 **Find each measure.**

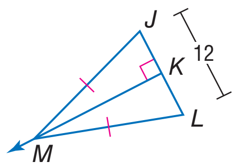
9. *NP*



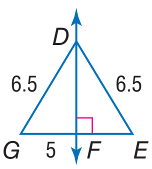
10. *PS*



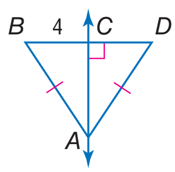
11. *KL*



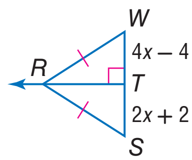
12. *EG*



13. *CD*

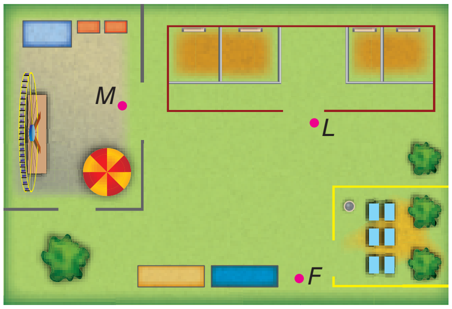


14. *SW*

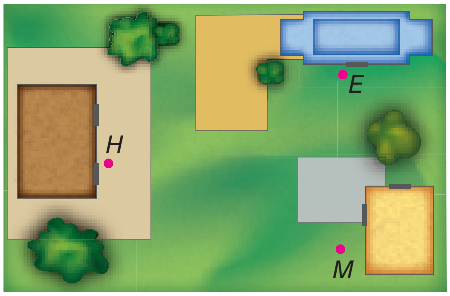


Example 2

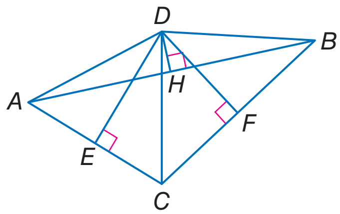
15. STATE FAIR The state fair has set up the location of the midway, livestock competition, and food vendors. The fair planners decide that they want to locate the portable restrooms the same distance from each location. Copy the positions of points M, L, and F. Then find the location for the restrooms and label it R.



16. SCHOOL A school system has built an elementary, middle, and high school at the locations shown in the diagram. Copy the positions of points E, M, and H. Then find the location for the bus yard B that will service these schools so that it is the same distance from each school.



Point D is the circumcenter of $\triangle ABC$. List any segment(s) congruent to each segment.



17. $\overline{AD}$

18. $\overline{BF}$

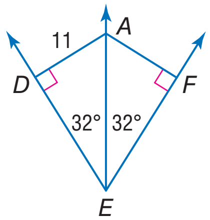
19. $\overline{AH}$

20. $\overline{DC}$

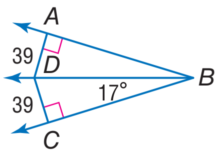
Example 3

Find each measure.

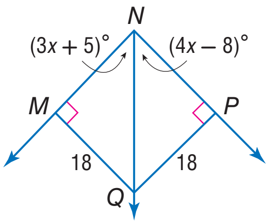
21. AF



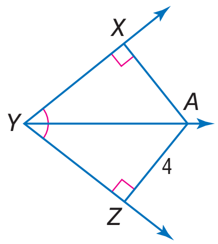
22. $m\angle DBA$



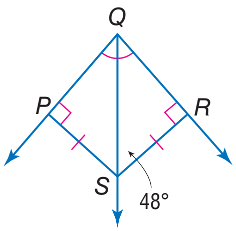
23 $m\angle PNM$



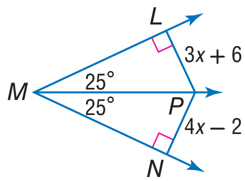
24. XA



25. $m\angle PQS$

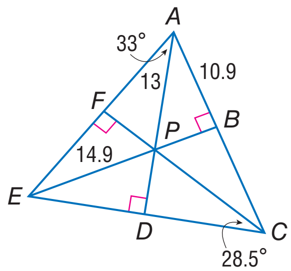


26. PN



Example 4

CCSS SENSE-MAKING **Point P is the incenter of $\triangle AEC$. Find each measure below.**



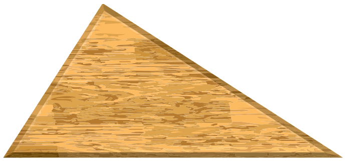
27. PB

28. DE

29. $m\angle DAC$

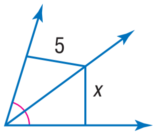
30. $m\angle DEP$

31 **INTERIOR DESIGN** You want to place a centerpiece on a corner table so that it is located the same distance from each edge of the table. Make a sketch to show where you should place the centerpiece. Explain your reasoning.

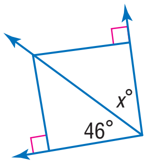


Determine whether there is enough information given in each diagram to find the value of x. Explain your reasoning.

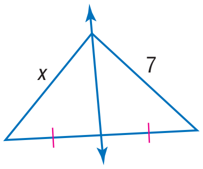
32.



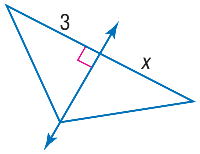
33.



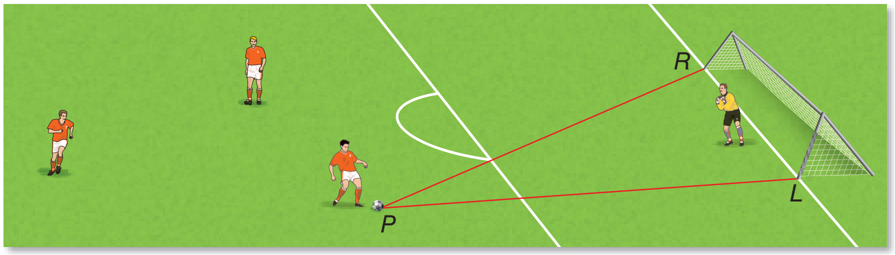
34.



35.



36. **SOCCER** A soccer player P is approaching the opposing team's goal as shown in the diagram. To make the goal, the player must kick the ball between the goal posts at L and R. The goalkeeper faces the kicker. He then tries to stand so that if he needs to dive to stop a shot, he is as far from the left-hand side of the shot angle as the right-hand side.



a. Describe where the goalkeeper should stand. Explain your reasoning.

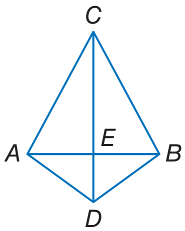
b. Copy $\triangle PRL$. Use a compass and a straightedge to locate a point G where the goalkeeper should stand.

c. If the ball is kicked so it follows the path from P to R, construct the shortest path the goalkeeper should take to block the shot. Explain your reasoning.

PROOF **Write a two-column proof.**

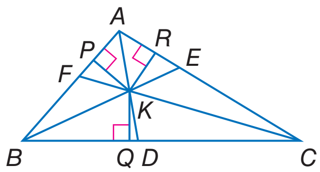
37. Theorem 5.2

Given: $\overline{CA} \cong \overline{CB}$, $\overline{AD} \cong \overline{BD}$
Prove: C and D are on the perpendicular bisector of $\overline{AB}$.



38. Theorem 5.6

Given: $\triangle ABC$, angle bisectors $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$
$\overline{KP} \perp \overline{AB}$, $\overline{KQ} \perp \overline{BC}$,
$\overline{KR} \perp \overline{AC}$
Prove: $KP = KQ = KR$



CCSS ARGUMENTS **Write a paragraph proof of each theorem.**

39. Theorem 5.1

40. Theorem 5.5

COORDINATE GEOMETRY **Write an equation in slope-intercept form for the perpendicular bisector of the segment with the given endpoints. Justify your answer.**

41. $A(-3, 1)$ and $B(4, 3)$

42. $C(-4, 5)$ and $D(2, -2)$

43. **PROOF** Write a two-column proof of Theorem 5.4.

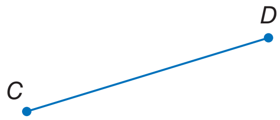
44. **GRAPHIC DESIGN** Mykia is designing a pennant for her school. She wants to put a picture of the school mascot inside a circle on the pennant. Copy the outline of the pennant and locate the point where the center of the circle should be to create the largest circle possible. Justify your drawing.

COORDINATE GEOMETRY **Find the coordinates of the circumcenter of the triangle with the given vertices. Explain.**

45. $A(0, 0)$, $B(0, 6)$, $C(10, 0)$

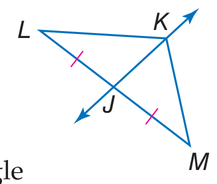
46. $J(5, 0)$, $K(5, -8)$, $L(0, 0)$

47. **LOCUS** Consider $\overline{CD}$. Describe the set of all points in space that are equidistant from C and D.



H.O.T. Problems Use Higher-Order Thinking Skills

48. **ERROR ANALYSIS** Claudio says that from the information supplied in the diagram, he can conclude that K is on the perpendicular bisector of $\overline{LM}$. Caitlyn disagrees. Is either of them correct? Explain your reasoning.



49. **OPEN ENDED** Draw a triangle with an incenter located inside the triangle but a circumcenter located outside. Justify your drawing by using a straightedge and a compass to find both points of concurrency.

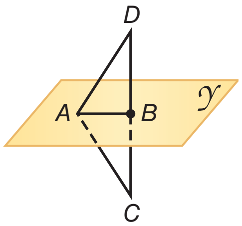
CCSS **ARGUMENTS** **Determine whether each statement is *sometimes*, *always*, or *never* true. Justify your reasoning using a counterexample or proof.**

50. The angle bisectors of a triangle intersect at a point that is equidistant from the vertices of the triangle.

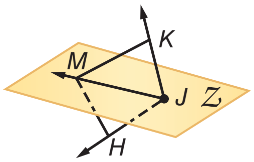
51. In an isosceles triangle, the perpendicular bisector of the base is also the angle bisector of the opposite vertex.

CHALLENGE **Write a two-column proof for each of the following.**

52. **Given:** Plane $\mathcal{Y}$ is a perpendicular bisector of $\overline{DC}$.
Prove: $\angle ADB \cong \angle ACB$



53. **Given:** Plane $\mathcal{Z}$ is an angle bisector of $\angle KJH$, $\overline{KJ} \cong \overline{HJ}$
Prove: $\overline{MH} \cong \overline{MK}$



54. **WRITING IN MATH** Compare and contrast the perpendicular bisectors and angle bisectors of a triangle. How are they alike? How are they different? Be sure to compare their points of concurrency.

Standardized Test Practice

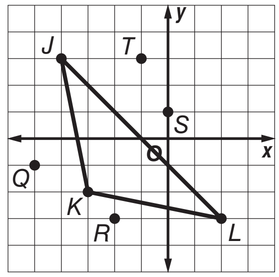
55. ALGEBRA An object is projected straight upward with initial velocity v meters per second from an initial height of s meters. The height h in meters of the object after t seconds is given by $h = -10t^2 + vt + s$. Sherise is standing at the edge of a balcony 54 meters above the ground and throws a ball straight up with an initial velocity of 12 meters per second. After how many seconds will it hit the ground?

A 3 seconds
B 4 seconds
C 6 seconds
D 9 seconds

56. SAT/ACT For $x \neq -3$, $\dfrac{3x + 9}{x + 3} =$

F $x + 12$
G $x + 9$
H $x + 3$
J x
K 3

57. A line drawn through which of the following points would be a perpendicular bisector of $\triangle JKL$?



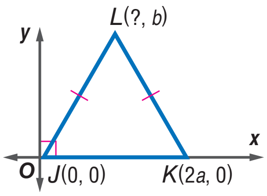
A T and K
B L and Q
C J and R
D S and K

58. SHORT RESPONSE Write an equation in slope-intercept form that describes the line containing the points $(-1, 0)$ and $(2, 4)$.

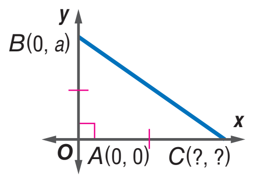
Spiral Review

Name the missing coordinate(s) of each triangle. (Lesson 4-8)

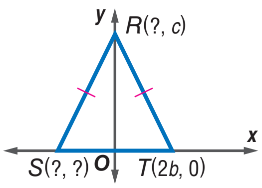
59.



60.



61.



COORDINATE GEOMETRY Graph each pair of triangles with the given vertices. Then identify the transformation and verify that it is a congruence transformation. (Lesson 4-7)

62. $A(-2, 4)$, $B(-2, -2)$, $C(4, 1)$;
$R(12, 4)$, $S(12, -2)$, $T(6, 1)$

63. $J(-3, 3)$, $K(-3, 1)$, $L(1, 1)$;
$X(-3, -1)$, $Y(-3, -3)$, $Z(1, -3)$

Find the distance from the line to the given point. (Lesson 3-6)

64. $y = 5$, $(-2, 4)$

65. $y = 2x + 2$, $(-1, -5)$

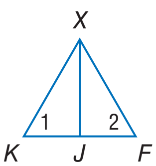
66. $2x - 3y = -9$, $(2, 0)$

67. AUDIO ENGINEERING A studio engineer charges a flat fee of \$450 for equipment rental and \$42 an hour for recording and mixing time. Write the equation that shows the cost to hire the studio engineer as a function of time. How much would it cost to hire the studio engineer for 17 hours? (Lesson 3-4)

Skills Review

PROOF Write a two-column proof for each of the following.

68. Given: $\triangle XKF$ is equilateral.
$\overline{XJ}$ bisects $\angle X$.
Prove: J is the midpoint of $\overline{KF}$.



69. Given: $\triangle MLP$ is isosceles.
N is the midpoint of $\overline{MP}$.
Prove: $\overline{LN} \perp \overline{MP}$