LESSON 5-2

# Medians and Altitudes of Triangles

**Then**

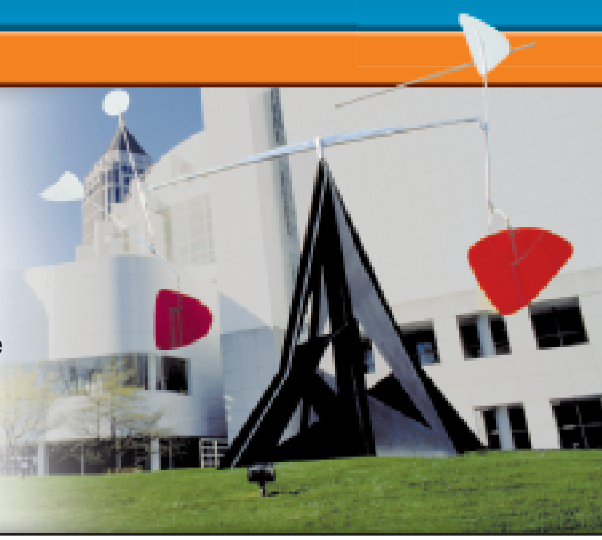
- You identified and used perpendicular and angle bisectors in triangles.

**Now**

1. Identify and use medians in triangles.
2. Identify and use altitudes in triangles.

**Why?**

- A mobile is a *kinetic* or moving sculpture that uses the principles of balance and equilibrium. Simple mobiles consist of several rods attached by strings from which objects of varying weights hang. The hanging objects balance each other and can rotate freely. To ensure that a triangle in a mobile hangs parallel to the ground, artists have to find the triangle's balancing point.



James Randklev/Stone/Getty Images

**NewVocabulary**

median
centroid
altitude
orthocenter

**Common Core State Standards**

**Content Standards**

G.CO.10 Prove theorems about triangles.

G.MG.3 Apply geometric methods to solve problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

**Mathematical Practices**

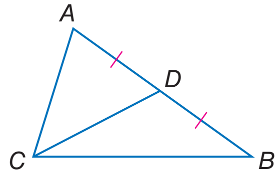
6 Attend to precision.

3 Construct viable arguments and critique the reasoning of others.

## 1 Medians

A **median** of a triangle is a segment with endpoints being a vertex of a triangle and the midpoint of the opposite side.

Every triangle has three medians that are concurrent. The point of concurrency of the medians of a triangle is called the **centroid** and is always inside the triangle.



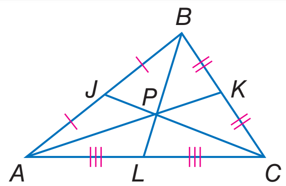

$\overline{CD}$ is a median of $\triangle ABC$.

### Theorem 5.7 Centroid Theorem

The medians of a triangle intersect at a point called the centroid that is two thirds of the distance from each vertex to the midpoint of the opposite side.

**Example** If $P$ is the centroid of $\triangle ABC$, then

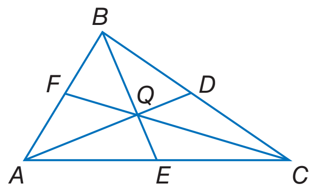
$AP = \frac{2}{3}AK$, $BP = \frac{2}{3}BL$, and $CP = \frac{2}{3}CJ$.




You will prove Theorem 5.7 in Exercise 36.

### Example 1 Use the Centroid Theorem

**In $\triangle ABC$, $Q$ is the centroid and $BE = 9$. Find $BQ$ and $QE$.**




| | |
|---|---|
| $BQ = \frac{2}{3}BE$ | Centroid Theorem |
| $= \frac{2}{3}(9)$ or 6 | $BE = 9$ |
| $BQ + QE = 9$ | Segment Addition |
| $6 + QE = 9$ | $BQ = 6$ |
| $QE = 3$ | Subtract 6 from each side. |

**GuidedPractice** **In $\triangle ABC$ above, $FC = 15$. Find each length.**

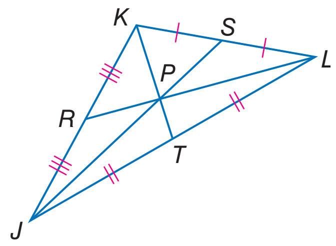
**1A.** $FQ$ **1B.** $QC$

**StudyTip**

**CCSS Reasoning** In Example 2, you can also use number sense to find *KP*. Since $KP = \frac{2}{3}KT$, $PT = \frac{1}{3}KT$ and $KP = 2PT$. Therefore, if $PT = 2$, then $KP = 2(2)$ or 4.

## Example 2 Use the Centroid Theorem

**In $\triangle JKL$, $PT = 2$. Find $KP$.**




Since $\overline{JR} \cong \overline{RK}$, $R$ is the midpoint of $\overline{JK}$ and $\overline{LR}$ is a median of $\triangle JKL$. Likewise, $S$ and $T$ are the midpoints of $\overline{KL}$ and $\overline{LJ}$ respectively, so $\overline{JS}$ and $\overline{KT}$ are also medians of $\triangle JKL$. Therefore, point $P$ is the centroid of $\triangle JKL$.

| | |
|---|---|
| $KP = \frac{2}{3}KT$ | Centroid Theorem |
| $KP = \frac{2}{3}(KP + PT)$ | Segment Addition and Substitution |
| $KP = \frac{2}{3}(KP + 2)$ | $PT = 2$ |
| $KP = \frac{2}{3}KP + \frac{4}{3}$ | Distributive Property |
| $\frac{1}{3}KP = \frac{4}{3}$ | Subtract $\frac{2}{3}KP$ from each side. |
| $KP = 4$ | Multiply each side by 3. |

### GuidedPractice

**In $\triangle JKL$ above, $RP = 3.5$ and $JP = 9$. Find each measure.**

**2A.** $PL$

**2B.** $PS$

All polygons have a balance point or centroid. The centroid is also the balancing point or *center of gravity* for a triangular region. The center of gravity is the point at which the region is stable under the influence of gravity.

## Real-World Example 3 Find the Centroid on Coordinate Plane

**PERFORMANCE ART** **A performance artist plans to balance triangular pieces of metal during her next act. When one such triangle is placed on the coordinate plane, its vertices are located at (1, 10), (5, 0), and (9, 5). What are the coordinates of the point where the artist should support the triangle so that it will balance?**

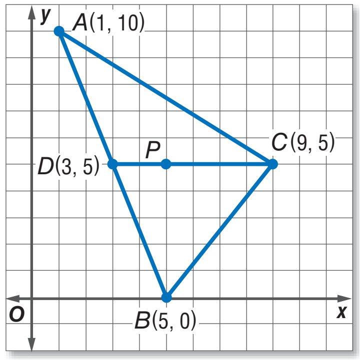
**Understand** You need to find the centroid of the triangle with the given coordinates. This is the point at which the triangle will balance.

**Plan** Graph and label the triangle with vertices $A(1, 10)$, $B(5, 0)$, and $C(9, 5)$. Since the centroid is the point of concurrency of the medians of a triangle, use the Midpoint Theorem to find the midpoint of one of the sides of the triangle. The centroid is two-thirds the distance from the opposite vertex to that midpoint.

**Math History Link**

**Pierre de Fermat**
**(1601–1665)** Another triangle center is the Fermat point, which minimizes the sum of the distances from the three vertices. Fermat is one of the best-known mathematicians for writing proofs.

**Solve** Graph $\triangle ABC$.




Find the midpoint $D$ of side $\overline{AB}$ with endpoints $A(1, 10)$ and $B(5, 0)$.

$$D\left(\frac{1+5}{2}, \frac{10+0}{2}\right) = D(3, 5)$$

Graph point $D$. Notice that $\overline{DC}$ is a horizontal line. The distance from $D(3, 5)$ to $C(9, 5)$ is $9 - 3$ or 6 units.

If $P$ is the centroid of $\triangle ABC$, then $PC = \frac{2}{3}DC$. So the centroid is $\frac{2}{3}(6)$ or 4 units to the left of $C$. The coordinates of $P$ are $(9 - 4, 5)$ or $(5, 5)$.

The performer should balance the triangle at the point $(5, 5)$.

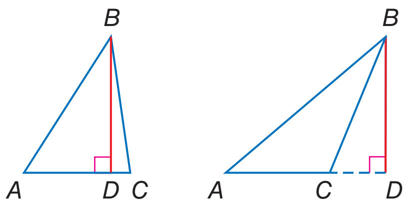
**Check** Use a different median to check your answer. The midpoint $F$ of side $\overline{AC}$ is $F\left(\frac{1+9}{2}, \frac{10+5}{2}\right)$ or $F(5, 7.5)$. $\overline{BF}$ is a vertical line, so the distance from $B$ to $F$ is $7.5 - 0$ or 7.5. $\overline{PB} = \frac{2}{3}(7.5)$ or 5, so $P$ is 5 units up from $B$. The coordinates of $P$ are $(5, 0 + 5)$ or $(5, 5)$. ✓

**Guided Practice**

**3.** A second triangle has vertices at (0, 4), (6, 11.5), and (12, 1). What are the coordinates of the point where the artist should support the triangle so that it will balance? Explain your reasoning.

## 2 Altitudes

An **altitude** of a triangle is a segment from a vertex to the line containing the opposite side and perpendicular to the line containing that side. An altitude can lie in the interior, exterior, or on the side of a triangle.




$\overline{BD}$ is an altitude from $B$ to $\overline{AC}$.

**Reading Math**

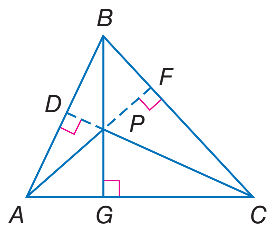
**Height of a Triangle** The length of an altitude is known as the *height* of the triangle. The height of a triangle is used to calculate the triangle's area.

Every triangle has three altitudes. If extended, the altitudes of a triangle intersect in a common point.

### Key Concept Orthocenter

The lines containing the altitudes of a triangle are concurrent, intersecting at a point called the **orthocenter**.

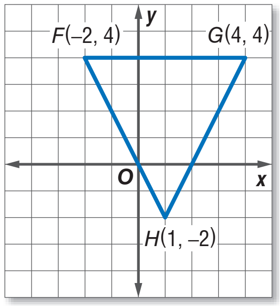
**Example** The lines containing altitudes $\overline{AF}$, $\overline{CD}$, and $\overline{BG}$ intersect at $P$, the orthocenter of $\triangle ABC$.



## Example 4 Find the Orthocenter on a Coordinate Plane

**COORDINATE GEOMETRY** **The vertices of $\triangle FGH$ are $F(-2, 4)$, $G(4, 4)$, and $H(1, -2)$. Find the coordinates of the orthocenter of $\triangle FGH$.**

**Step 1** Graph $\triangle FGH$. To find the orthocenter, find the point where two of the three altitudes intersect.




**Step 2** Find an equation of the altitude from $F$ to $\overline{GH}$. The slope of $\overline{GH}$ is $\frac{4-(-2)}{4-1}$ or 2, so the slope of the altitude, which is perpendicular to $\overline{GH}$, is $-\frac{1}{2}$.

| | |
|---|---|
| $y - y_1 = m(x - x_1)$ | Point-slope form |
| $y - 4 = -\frac{1}{2}[x - (-2)]$ | $m = -\frac{1}{2}$ and $(x_1, y_1) = F(-2, 4)$. |
| $y - 4 = -\frac{1}{2}(x + 2)$ | Simplify. |
| $y - 4 = -\frac{1}{2}x - 1$ | Distributive Property |
| $y = -\frac{1}{2}x + 3$ | Add 4 to each side. |

Find an equation of the altitude from $G$ to $\overline{FH}$. The slope of $\overline{FH}$ is $\frac{-2-4}{1-(-2)}$ or $-2$, so the slope of the altitude is $\frac{1}{2}$.

| | |
|---|---|
| $y - y_1 = m(x - x_1)$ | Point-slope form |
| $y - 4 = \frac{1}{2}(x - 4)$ | $m = \frac{1}{2}$ and $(x_1, y_1) = G(4, 4)$ |
| $y - 4 = \frac{1}{2}x - 2$ | Distributive Property |
| $y = \frac{1}{2}x + 2$ | Add 4 to each side. |

**Step 3** Solve the resulting system of equations $\begin{cases} y = -\frac{1}{2}x + 3 \\ y = \frac{1}{2}x + 2 \end{cases}$ to find the point of intersection of the altitudes.

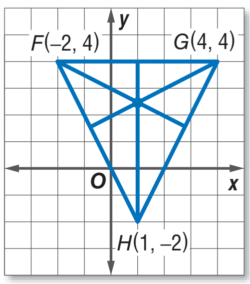
Adding the two equations to eliminate $x$ results in $2y = 5$ or $y = \frac{5}{2}$.

| | |
|---|---|
| $y = \frac{1}{2}x + 2$ | Equation of altitude from $G$ |
| $\frac{5}{2} = \frac{1}{2}x + 2$ | $y = \frac{5}{2}$ |
| $\frac{1}{2} = \frac{1}{2}x$ | Subtract $\frac{4}{2}$ or 2 from each side. |
| $1 = x$ | Multiply each side by 2. |

The coordinates of the orthocenter of $\triangle JKL$ are $\left(1, \frac{5}{2}\right)$ or $\left(1, 2\frac{1}{2}\right)$.

### StudyTip

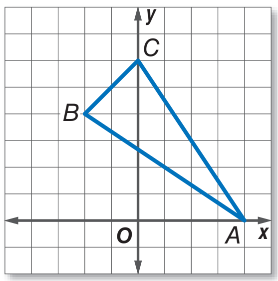
**Check for Reasonableness** Use the corner of a sheet of paper to draw the altitudes of each side of the triangle.



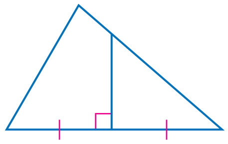
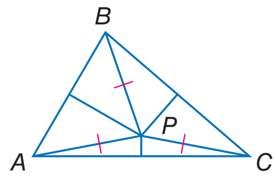
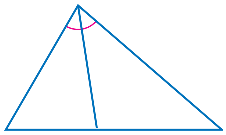
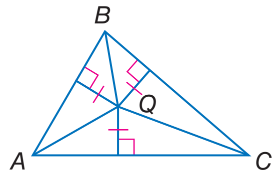
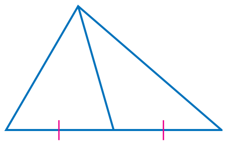
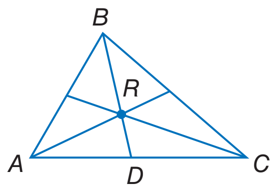
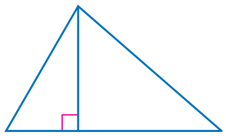
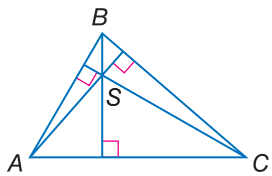

The intersection is located at approximately $\left(1, 2\frac{1}{2}\right)$, so the answer is reasonable.

### GuidedPractice

**4.** Find the coordinates of the orthocenter of $\triangle ABC$ graphed at the right.



## ConceptSummary Special Segments and Points in Triangles

| Name | Example | Point of Concurrency | Special Property | Example |
|---|---|---|---|---|
| perpendicular bisector |  | circumcenter | The circumcenter *P* of $\triangle ABC$ is equidistant from each vertex. |   |
| angle bisector |  | incenter | The incenter *Q* of $\triangle ABC$ is equidistant from each side of the triangle. |   |
| median |  | centroid | The centroid *R* of $\triangle ABC$ is two thirds of the distance from each vertex to the midpoint of the opposite side. |   |
| altitude |  | orthocenter | The lines containing the altitudes of $\triangle ABC$ are concurrent at the orthocenter *S*. |   |

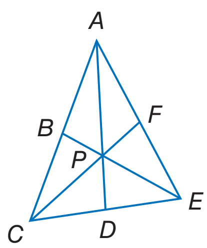
## Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

**Examples 1–2** **In $\triangle ACE$, $P$ is the centroid, $PF = 6$, and $AD = 15$. Find each measure.**

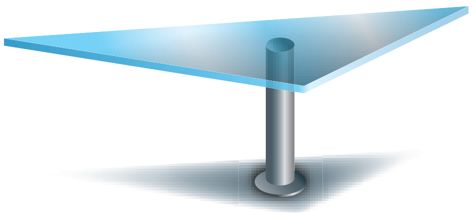
**1.** $PC$

**2.** $AP$




**Example 3**

**3. INTERIOR DESIGN** An interior designer is creating a custom coffee table for a client. The top of the table is a glass triangle that needs to balance on a single support. If the coordinates of the vertices of the triangle are at (3, 6), (5, 2), and (7, 10), at what point should the support be placed?



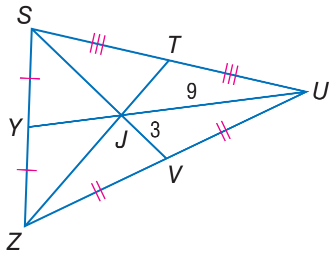
**Example 4**

**4. COORDINATE GEOMETRY** Find the coordinates of the orthocenter of $\triangle ABC$ with vertices $A(-3, 3)$, $B(-1, 7)$, and $C(3, 3)$.

## Practice and Problem Solving

Extra Practice is on page R5.

**Examples 1–2** **In $\triangle SZU$, $UJ = 9$, $VJ = 3$, and $ZT = 18$. Find each length.**

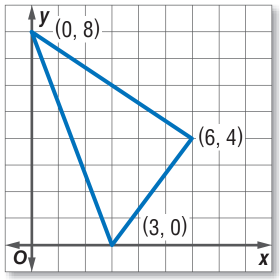



5. $YJ$
6. $SJ$
7. $YU$
8. $SV$
9. $JT$
10. $ZJ$

**Example 3** **COORDINATE GEOMETRY** **Find the coordinates of the centroid of each triangle with the given vertices.**

11. $A(-1, 11)$, $B(3, 1)$, $C(7, 6)$
12. $X(5, 7)$, $Y(9, -3)$, $Z(13, 2)$

13. **INTERIOR DESIGN** Emilia made a collage with pictures of her friends. She wants to hang the collage from the ceiling in her room so that it is parallel to the ceiling. A diagram of the collage is shown in the graph at the right. At what point should she place the string?

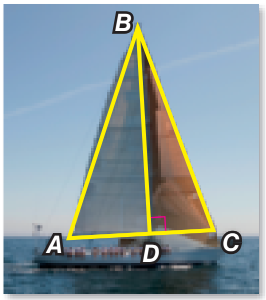



**Example 4** **COORDINATE GEOMETRY** **Find the coordinates of the orthocenter of each triangle with the given vertices.**

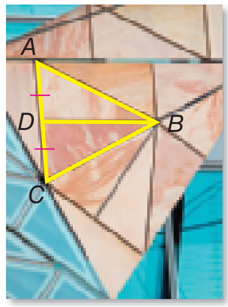
14. $J(3, -2)$, $K(5, 6)$, $L(9, -2)$
15. $R(-4, 8)$, $S(-1, 5)$, $T(5, 5)$

**Identify each segment $\overline{BD}$ as a(n) altitude, median, or perpendicular bisector.**

16.




17.

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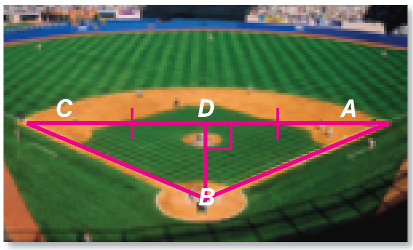
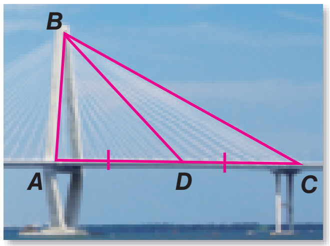
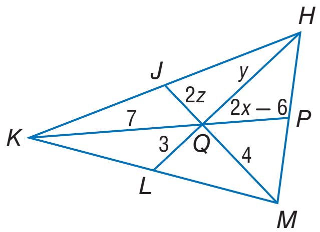



19.




20. **CCSS SENSE-MAKING** In the figure at the right, if $J$, $P$, and $L$ are the midpoints of $\overline{KH}$, $\overline{HM}$, and $\overline{MK}$, respectively, find $x$, $y$, and $z$.



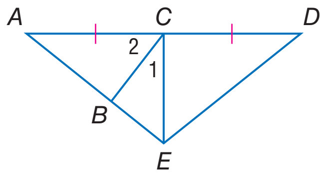
**Copy and complete each statement for $\triangle RST$ for medians $\overline{RM}$, $\overline{SL}$ and $\overline{TK}$, and centroid $J$.**

**21.** $SL = x(JL)$

**22.** $JT = x(TK)$

**23.** $JM = x(RJ)$

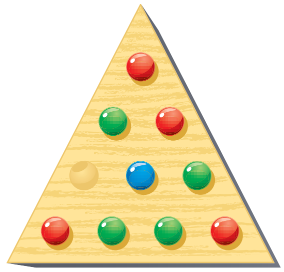
**ALGEBRA** **Use the figure at the right.**



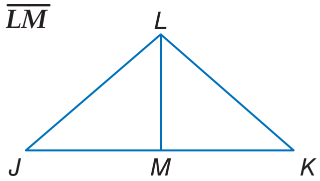

**24.** If $\overline{EC}$ is an altitude of $\triangle AED$, $m\angle 1 = 2x + 7$, and $m\angle 2 = 3x + 13$, find $m\angle 1$ and $m\angle 2$.

**25.** Find the value of $x$ if $AC = 4x - 3$, $DC = 2x + 9$, $m\angle ECA = 15x + 2$, and $\overline{EC}$ is a median of $\triangle AED$. Is $\overline{EC}$ also an altitude of $\triangle AED$? Explain.

**26.** **GAMES** The game board shown is shaped like an equilateral triangle and has indentations for game pieces. The game's objective is to remove pegs by jumping over them until there is only one peg left. Copy the game board's outline and determine which of the points of concurrency the blue peg represents: *circumcenter, incenter, centroid,* or *orthocenter*. Explain your reasoning.



**CCSS ARGUMENTS** **Use the given information to determine whether $\overline{LM}$ is a *perpendicular bisector, median,* and/or an *altitude* of $\triangle JKL$.**




**27.** $\overline{LM} \perp \overline{JK}$

**28.** $\triangle JLM \cong \triangle KLM$

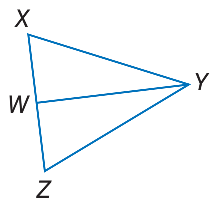
**29.** $\overline{JM} \cong \overline{KM}$

**30.** $\overline{LM} \perp \overline{JK}$ and $\overline{JL} \cong \overline{KL}$

**31.** **PROOF** Write a paragraph proof.

**Given:** $\triangle XYZ$ is isosceles. $\overline{WY}$ bisects $\angle Y$.

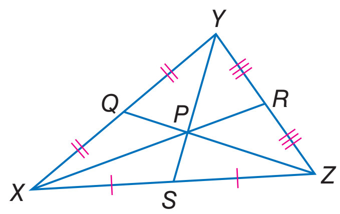
**Prove:** $\overline{WY}$ is a median.




**32.** **PROOF** Write an algebraic proof.

**Given:** $\triangle XYZ$ with medians $\overline{XR}$, $\overline{YS}$, $\overline{ZQ}$

**Prove:** $\frac{XP}{PR} = 2$



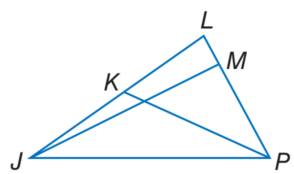

**33.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the location of the points of concurrency for any equilateral triangle.

**a.** **Concrete** Construct three different equilateral triangles on tracing paper and cut them out. Fold each triangle to locate the circumcenter, incenter, centroid, and orthocenter.

**b.** **Verbal** Make a conjecture about the relationships among the four points of concurrency of any equilateral triangle.

**c.** **Graphical** Position an equilateral triangle and its circumcenter, incenter, centroid, and orthocenter on the coordinate plane using variable coordinates. Determine the coordinates of each point of concurrency.

**ALGEBRA** **In $\triangle JLP$, $m\angle JMP = 3x - 6$, $JK = 3y - 2$, and $LK = 5y - 8$.**



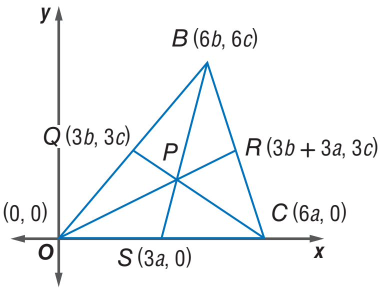

**34.** If $\overline{JM}$ is an altitude of $\triangle JLP$, find $x$.

**35.** Find $LK$ if $\overline{PK}$ is a median.

**36.** **PROOF** **Write a coordinate proof to prove the Centroid Theorem.**

**Given:** $\triangle ABC$, medians $\overline{AR}$, $\overline{BS}$, and $\overline{CQ}$

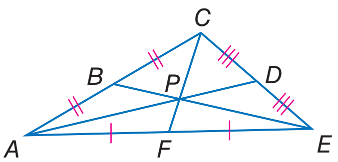
**Prove:** The medians intersect at point $P$ and $P$ is two thirds of the distance from each vertex to the midpoint of the opposite side.




(*Hint*: First, find the equations of the lines containing the medians. Then find the coordinates of point $P$ and show that all three medians intersect at point $P$. Next, use the Distance Formula and multiplication to show $AP = \frac{2}{3}AR$, $BP = \frac{2}{3}BS$, and $CP = \frac{2}{3}CQ$.)

## H.O.T. Problems Use Higher-Order Thinking Skills

**37** **ERROR ANALYSIS** Based on the figure at the right, Luke says that $\frac{2}{3}AP = AD$. Kareem disagrees. Is either of them correct? Explain your reasoning.



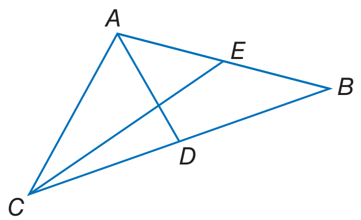

**38.** CCSS **ARGUMENTS** Determine whether the following statement is *true* or *false*. If true, explain your reasoning. If false, provide a counterexample.

*The orthocenter of a right triangle is always located at the vertex of the right angle.*

**39.** **CHALLENGE** $\triangle ABC$ has vertices $A(-3, 3)$, $B(2, 5)$, and $C(4, -3)$. What are the coordinates of the centroid of $\triangle ABC$? Explain the process you used to reach your conclusion.

**40.** **WRITING IN MATH** Compare and contrast the perpendicular bisectors, medians, and altitudes of a triangle.

**41.** **CHALLENGE** In the figure at the right, segments $\overline{AD}$ and $\overline{CE}$ are medians of $\triangle ACB$, $\overline{AD} \perp \overline{CE}$, $AB = 10$, and $CE = 9$. Find $CA$.




**42.** **OPEN ENDED** In this problem, you will investigate the relationships among three points of concurrency in a triangle.

**a.** Draw an acute triangle and find the circumcenter, centroid, and orthocenter.

**b.** Draw an obtuse triangle and find the circumcenter, centroid, and orthocenter.

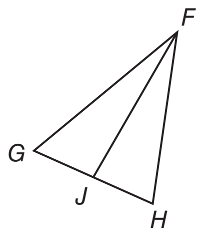
**c.** Draw a right triangle and find the circumcenter, centroid, and orthocenter.

**d.** Make a conjecture about the relationships among the circumcenter, centroid, and orthocenter.

**43.** **WRITING IN MATH** Use area to explain why the centroid of a triangle is its center of gravity. Then use this explanation to describe the location for the balancing point for a rectangle.

## Standardized Test Practice

**44.** In the figure below, $\overline{GJ} \cong \overline{HJ}$. Which must be true?




**A** $\overline{FJ}$ is an altitude of $\triangle FGH$.

**B** $\overline{FJ}$ is an angle bisector of $\triangle FGH$.

**C** $\overline{FJ}$ is a median of $\triangle FGH$.

**D** $\overline{FJ}$ is a perpendicular bisector of $\triangle FGH$.

**45. GRIDDED RESPONSE** What is the $x$-intercept of the graph of $4x - 6y = 12$?

**46. ALGEBRA** Four students have volunteered to fold pamphlets for a local community action group. Which student is the fastest?

| Student | Folding Speed |
|---|---|
| Neiva | 1 page every 3 seconds |
| Sarah | 2 pages every 10 seconds |
| Quinn | 30 pages per minute |
| Deron | 45 pages in 2 minutes |

**F** Deron  **H** Quinn
**G** Neiva  **J** Sarah

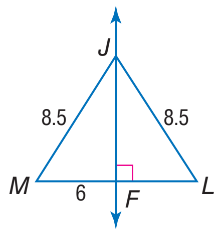
**47. SAT/ACT** 80 percent of 42 is what percent of 16?

**A** 240  **D** 50
**B** 210  **E** 30
**C** 150

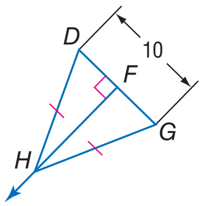
## Spiral Review

**Find each measure.** (Lesson 5-1)

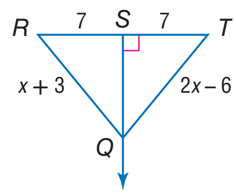
**48.** $LM$




**49.** $DF$




**50.** $TQ$




**Position and label each triangle on the coordinate plane.** (Lesson 4-8)

**51.** right $\triangle XYZ$ with hypotenuse $\overline{XZ}$, $ZY$ is twice $XY$, and $\overline{XY}$ is $b$ units long

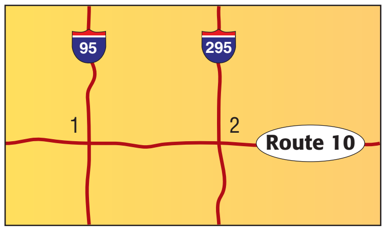
**52.** isosceles $\triangle QRT$ with base $\overline{QR}$ that is $b$ units long

**Determine whether $\overleftrightarrow{RS}$ and $\overleftrightarrow{JK}$ are *parallel*, *perpendicular*, or *neither*. Graph each line to verify your answer.** (Lesson 3-3)

**53.** $R(5, -4)$, $S(10, 0)$, $J(9, -8)$, $K(5, -13)$

**54.** $R(1, 1)$, $S(9, 8)$, $J(-6, 1)$, $K(2, 8)$

**55. HIGHWAYS** Near the city of Hopewell, Virginia, Route 10 runs perpendicular to Interstate 95 and Interstate 295. Show that the angles at the intersections of Route 10 with Interstate 95 and Interstate 295 are congruent. (Lesson 2-8)




## Skills Review

**PROOF** **Write a flow proof of the Exterior Angle Theorem.**

**56. Given:** $\triangle XYZ$

**Prove:** $m\angle X + m\angle Z = m\angle 1$