LESSON 5-4

Indirect Proof

Then

- You wrote paragraph, two-column, and flow proofs.

Now

1. Write indirect algebraic proofs.
2. Write indirect geometric proofs.

Why?

- **Matthew:** "I'm almost positive Friday is not a teacher work day, but I can't prove it."

Kim: "Let's assume that Friday *is* a teacher work day. What day is our next Geometry test?"

Ana: "Hmmm . . . according to the syllabus, it's this Friday. But we don't have tests on teacher work days—we're not in school."

Jamal: "Exactly—so that proves it! This Friday can't be a teacher work day."

NewVocabulary
indirect reasoning
indirect proof
proof by contradiction

Common Core State Standards

Content Standards
G.CO.10 Prove theorems about triangles.

Mathematical Practices
3 Construct viable arguments and critique the reasoning of others.
2 Reason abstractly and quantitatively.

Digital Vision/Punchstock

1 Indirect Algebraic Proof

The proofs you have written have been *direct proofs*—you started with a true hypothesis and proved that the conclusion was true. In the example above, the students used **indirect reasoning**, by assuming that a conclusion was false and then showing that this assumption led to a contradiction.

In an **indirect proof** or **proof by contradiction**, you temporarily assume that what you are trying to prove is false. By showing this assumption to be logically impossible, you prove your assumption false and the original conclusion true. Sometimes this is called *proof by negation*.

KeyConcept How to Write an Indirect Proof

Step 1 Identify the conclusion you are asked to prove. Make the assumption that this conclusion is false by assuming that the opposite is true.

Step 2 Use logical reasoning to show that this assumption leads to a contradiction of the hypothesis, or some other fact, such as a definition, postulate, theorem, or corollary.

Step 3 Point out that since the assumption leads to a contradiction, the original conclusion, what you were asked to prove, must be true.

Example 1 State the Assumption for Starting an Indirect Proof

State the assumption necessary to start an indirect proof of each statement.

a. If 6 is a factor of n, then 2 is a factor of n.

The conclusion of the conditional statement is *2 is a factor of n*. The negation of the conclusion is *2 is not a factor of n*.

b. $\angle 3$ is an obtuse angle.

If *$\angle 3$ is an obtuse angle* is false, then *$\angle 3$ is not an obtuse angle* must be true.

GuidedPractice

1A. $x > 5$

1B. $\triangle XYZ$ is an equilateral triangle.

Indirect proofs can be used to prove algebraic concepts.

Example 2 Write an Indirect Algebraic Proof

Write an indirect proof to show that if $-3x + 4 > 16$, then $x < -4$.

Given: $-3x + 4 > 16$

Prove: $x < -4$

Step 1 **Indirect Proof:**
The negation of $x < -4$ is $x \geq -4$. So, assume that $x > -4$ or $x = -4$ is true.

Step 2 Make a table with several possibilities for x assuming $x > -4$ or $x = -4$.

x	−4	−3	−2	−1	0
$-3x + 4$	16	13	10	7	4

When $x > -4$, $-3x + 4 < 16$ and when $x = -4$, $-3x + 4 = 16$.

Step 3 In both cases, the assumption leads to the contradiction of the given information that $-3x + 4 > 16$. Therefore, the assumption that $x \geq -4$ must be false, so the original conclusion that $x < -4$ must be true.

GuidedPractice

Write an indirect proof of each statement.

2A. If $7x > 56$, then $x > 8$.

2B. If $-c$ is positive, then c is negative.

ReadingMath

Contradiction
A contradiction is a principle of logic stating that an assumption cannot be both *A* and the opposite of *A* at the same time.

Indirect reasoning and proof can be used in everyday situations.

Real-World Example 3 Indirect Algebraic Proof

PROM COSTS **Javier asked his friend Christopher the cost of his meal and his date's meal when he went to dinner for prom. Christopher could not remember the individual costs, but he did remember that the total bill, not including tip, was over \$60. Use indirect reasoning to show that at least one of the meals cost more than \$30.**

Let the cost of one meal be x and the cost of the other meal be y.

Step 1 **Given:** $x + y > 60$

Prove: $x > 30$ or $y > 30$

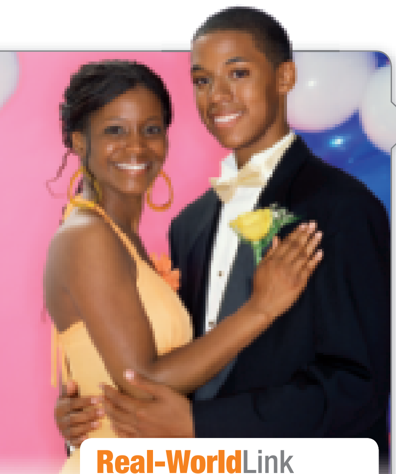
Indirect Proof:
Assume that $x \leq 30$ and $y \leq 30$.

Step 2 If $x \leq 30$ and $y \leq 30$, then $x + y \leq 30 + 30$ or $x + y \leq 60$. This is a contradiction because we know that $x + y > 60$.

Step 3 Since the assumption that $x \leq 30$ and $y \leq 30$ leads to a contradiction of a known fact, the assumption must be false. Therefore, the conclusion that $x > 30$ or $y > 30$ must be true. Thus, at least one of the meals had to cost more than \$30.

GuidedPractice

3. **TRAVEL** Cleavon traveled over 360 miles on his trip, making just two stops. Use indirect reasoning to prove that he traveled more than 120 miles on one leg of his trip.



Real-WorldLink

\$100–\$300 the range in price of a girl's prom dress

\$75–\$125 the range in cost for a tuxedo rental

around \$150 the cost of a fancy dinner for two

\$100–\$200 the range in cost of prom tickets per couple

Source: PromSpot

Stockbyte/Getty Images

Indirect proofs are often used to prove concepts in number theory. In such proofs, it is helpful to remember that you can represent an even number with the expression $2k$ and an odd number with the expression $2k + 1$ for any integer k.

PT

Example 4 Indirect Proofs in Number Theory

Write an indirect proof to show that if $x + 2$ is an even integer, then x is an even integer.

Step 1 **Given:** $x + 2$ is an even integer.

Prove: x is an even integer.

Indirect Proof:
Assume that x is an odd integer. This means that $x = 2k + 1$ for some integer k.

Step 2

$x + 2 = (2k + 1) + 2$	Substitution of assumption
$= (2k + 2) + 1$	Commutative Property
$= 2(k + 1) + 1$	Distributive Property

Now determine whether $2(k + 1) + 1$ is an even or odd integer. Since k is an integer, $k + 1$ is also an integer. Let m represent the integer $k + 1$.

$2(k + 1) + 1 = 2m + 1$ Substitution

So, $x + 2$ can be represented by $2m + 1$, where m is an integer. But this representation means that $x + 2$ is an odd integer, which contradicts the given statement that $x + 2$ is an even integer.

Step 3 Since the assumption that x is an odd integer leads to a contradiction of the given statement, the original conclusion that x is an even integer must be true.

WatchOut!

CCSS Arguments Proof by contradiction and using a counterexample are not the same. A counterexample helps you disprove a conjecture. It cannot be used to prove a conjecture.

GuidedPractice

4. Write an indirect proof to show that if the square of an integer is odd, then the integer is odd.

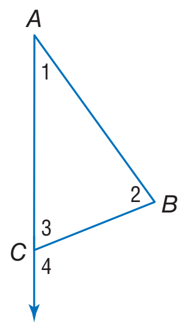
2 Indirect Proof with Geometry

Indirect reasoning can be used to prove statements in geometry, such as the Exterior Angle Inequality Theorem.

PT

Example 5 Geometry Proof

If an angle is an exterior angle of a triangle, prove that its measure is greater than the measure of either of its corresponding remote interior angles.



Step 1 Draw a diagram of this situation. Then identify what you are given and what you are asked to prove.

Given: $\angle 4$ is an exterior angle of $\triangle ABC$.

Prove: $m\angle 4 > m\angle 1$ and $m\angle 4 > m\angle 2$.

Indirect Proof:
Assume that $m\angle 4 \not> m\angle 1$ or $m\angle 4 \not> m\angle 2$.
In other words, $m\angle 4 \leq m\angle 1$ or $m\angle 4 \leq m\angle 2$.

(continued on the next page)

Step 2 You need only show that the assumption $m\angle 4 \leq m\angle 1$ leads to a contradiction. The argument for $m\angle 4 \leq m\angle 2$ follows the same reasoning.

$m\angle 4 \leq m\angle 1$ means that either $m\angle 4 = m\angle 1$ or $m\angle 4 < m\angle 1$.

Case 1 $m\angle 4 = m\angle 1$

$m\angle 4 = m\angle 1 + m\angle 2$ — Exterior Angle Theorem

$m\angle 4 = m\angle 4 + m\angle 2$ — Substitution

$0 = m\angle 2$ — Subtract $m\angle 4$ from each side.

This contradicts the fact that the measure of an angle is greater than 0, so $m\angle 4 \neq m\angle 1$.

Case 2 $m\angle 4 < m\angle 1$

By the Exterior Angle Theorem, $m\angle 4 = m\angle 1 + m\angle 2$. Since angle measures are positive, the definition of inequality implies that $m\angle 4 > m\angle 1$. This contradicts the assumption that $m\angle 4 < m\angle 1$.

Step 3 In both cases, the assumption leads to the contradiction of a theorem or definition. Therefore, the original conclusion that $m\angle 4 > m\angle 1$ and $m\angle 4 > m\angle 2$ must be true.

StudyTip

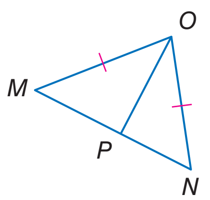
Recognizing Contradictions Remember that the contradiction in an indirect proof is not always of the given information or the assumption. It can be of a known fact or definition, such as in Case 1 of Example 5—the measure of an angle must be greater than 0.

GuidedPractice

5. Write an indirect proof.

Given: $\overline{MO} \cong \overline{ON}$, $\overline{MP} \ncong \overline{NP}$

Prove: $\angle MOP \ncong \angle NOP$



Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Example 1 **State the assumption you would make to start an indirect proof of each statement.**

1. $\overline{AB} \cong \overline{CD}$
2. $\triangle XYZ$ is a scalene triangle.
3. If $4x < 24$, then $x < 6$.
4. $\angle A$ is not a right angle.

Example 2 **Write an indirect proof of each statement.**

5. If $2x + 3 < 7$, then $x < 2$.
6. If $3x - 4 > 8$, then $x > 4$.

Example 3

7. **LACROSSE** Christina scored 13 points for her high school lacrosse team during the last six games. Prove that her average points per game was less than 3.

Example 4

8. Write an indirect proof to show that if $5x - 2$ is an odd integer, then x is an odd integer.

Example 5 **Write an indirect proof of each statement.**

9. The hypotenuse of a right triangle is the longest side.
10. If two angles are supplementary, then they both cannot be obtuse angles.

Practice and Problem Solving

Extra Practice is on page R5.

Example 1 **State the assumption you would make to start an indirect proof of each statement.**

11. If $2x > 16$, then $x > 8$.

12. $\angle 1$ and $\angle 2$ are not supplementary angles.

13. If two lines have the same slope, the lines are parallel.

14. If the consecutive interior angles formed by two lines and a transversal are supplementary, the lines are parallel.

15. If a triangle is not equilateral, the triangle is not equiangular.

16. An odd number is not divisible by 2.

Example 2 **Write an indirect proof of each statement.**

17. If $2x - 7 > -11$, then $x > -2$.

18. If $5x + 12 < -33$, then $x < -9$.

19. If $-3x + 4 < 7$, then $x > -1$.

20. If $-2x - 6 > 12$, then $x < -9$.

Example 3

21. **COMPUTER GAMES** Kwan-Yong bought two computer games for just over \$80 before tax. A few weeks later, his friend asked how much each game cost. Kwan-Yong could not remember the individual prices. Use indirect reasoning to show that at least one of the games cost more than \$40.

22. **FUNDRAISING** Jamila's school is having a Fall Carnival to raise money for a local charity. The cost of an adult ticket to the carnival is \$6 and the cost of a child's ticket is \$2.50. If 375 total tickets were sold and the profit was more than \$1460, prove that at least 150 adult tickets were sold.

Examples 4–5 CCSS **ARGUMENTS** **Write an indirect proof of each statement.**

23. **Given:** xy is an odd integer.

Prove: x and y are both odd integers.

24. **Given:** n^2 is even.

Prove: n^2 is divisible by 4.

25. **Given:** x is an odd number.

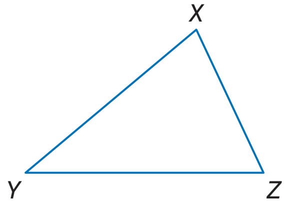
Prove: x is not divisible by 4.

26. **Given:** xy is an even integer.

Prove: x or y is an even integer.

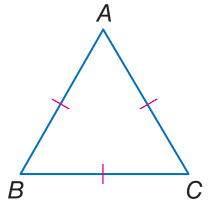
27. **Given:** $XZ > YZ$

Prove: $\angle X \not\cong \angle Y$



28. **Given:** $\triangle ABC$ is equilateral.

Prove: $\triangle ABC$ is equiangular.



29. In an isosceles triangle neither of the base angles can be a right angle.

30. A triangle can have only one right angle.

31. Write an indirect proof for Theorem 5.10.

32. Write an indirect proof to show that if $\frac{1}{b} < 0$, then b is negative.

33. **BASKETBALL** In basketball, there are three possible ways to score three points in a single possession. A player can make a basket from behind the three-point line, a player may be fouled while scoring a two-point shot and be allowed to shoot one free throw, or a player may be fouled behind the three-point line and be allowed to shoot three free throws. When Katsu left to get in the concession line, the score was 28 home team to 26 visiting team. When she returned, the score was 28 home team to 29 visiting team. Katsu concluded that a player on the visiting team had made a three-point basket. Prove or disprove her assumption using an indirect proof.

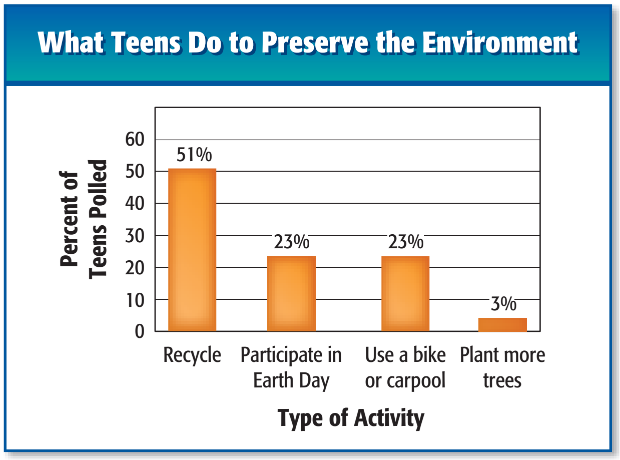
34. **GAMES** A computer game involves a knight on a quest for treasure. At the end of the journey, the knight approaches the two doors shown below.

A servant tells the knight that one of the signs is true and the other is false. Use indirect reasoning to determine which door the knight should choose. Explain your reasoning.

35 **SURVEYS** Luisa's local library conducted an online poll of teens to find out what activities teens participate in to preserve the environment. The results of the poll are shown in the graph.

a. Prove: *More than half of teens polled said that they recycle to preserve the environment.*

b. If 400 teens were polled, verify that 92 said that they participate in Earth Day.



36. **CCSS REASONING** James, Hector, and Mandy all have different color cars. Only one of the statements below is true. Use indirect reasoning to determine which statement is true. Explain.

(1) James has a red car.

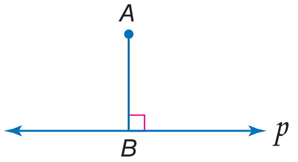
(2) Hector does not have a red car.

(3) Mandy does not have a blue car.

Determine whether each statement about the shortest distance between a point and a line or plane can be proved using a direct or indirect proof. Then write a proof of each statement.

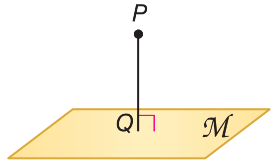
37. **Given:** $\overline{AB} \perp$ line p

Prove: $\overline{AB}$ is the shortest segment from A to line p.



38. **Given:** $\overline{PQ} \perp$ plane M

Prove: $\overline{PQ}$ is the shortest segment from P to plane M.



39. **NUMBER THEORY** In this problem, you will make and prove a conjecture about a number theory relationship.

a. Write an expression for *the sum of the cube of a number and three.*

b. Create a table that includes the value of the expression for 10 different values of n. Include both odd and even values of n.

c. Write a conjecture about n when the value of the expression is even.

d. Write an indirect proof of your conjecture.

H.O.T. Problems Use Higher-Order Thinking Skills

40. **WRITING IN MATH** Explain the procedure for writing an indirect proof.

41. **OPEN ENDED** Write a statement that can be proven using indirect proof. Include the indirect proof of your statement.

42. **CHALLENGE** If x is a rational number, then it can be represented by the quotient $\frac{a}{b}$ for some integers a and b, if $b \neq 0$. An irrational number cannot be represented by the quotient of two integers. Write an indirect proof to show that the product of a nonzero rational number and an irrational number is an irrational number.

43. **CCSS CRITIQUE** Amber and Raquel are trying to verify the following statement using indirect proof. Is either of them correct? Explain your reasoning.

If the sum of two numbers is even, then the numbers are even.

Amber	Raquel
The statement is true. If one of the numbers is even and the other number is zero, then the sum is even. Since the hypothesis is true even when the conclusion is false, the statement is true.	The statement is true. If the two numbers are odd, then the sum is even. Since the hypothesis is true when the conclusion is false, the statement is true.

44. **WRITING IN MATH** Refer to Exercise 8. Write the contrapositive of the statement and write a direct proof of the contrapositive. How are the direct proof of the contrapositive of the statement and the indirect proof of the statement related?

Standardized Test Practice

45. SHORT RESPONSE Write an equation in slope-intercept form to describe the line that passes through the point (5, 3) and is parallel to the line represented by the equation $-2x + y = -4$.

46. Statement: If $\angle A \cong \angle B$ and $\angle A$ is supplementary to $\angle C$, then $\angle B$ is supplementary to $\angle C$.

Dia is proving the statement above by contradiction. She began by assuming that $\angle B$ is not supplementary to $\angle C$. Which of the following definitions will Dia use to reach a contradiction?

- **A** definition of congruence
- **B** definition of a linear pair
- **C** definition of a right angle
- **D** definition of supplementary angles

47. List the angles of $\triangle MNO$ in order from smallest to largest if $MN = 9$, $NO = 7.5$, and $OM = 12$.

- **F** $\angle N, \angle O, \angle M$
- **G** $\angle O, \angle M, \angle N$
- **H** $\angle O, \angle N, \angle M$
- **J** $\angle M, \angle O, \angle N$

48. SAT/ACT If $b > a$, which of the following must be true?

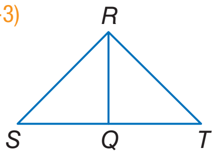
- **A** $-a > -b$
- **B** $3a > b$
- **C** $a^2 < b^2$
- **D** $a^2 < ab$
- **E** $-b > -a$

Spiral Review

49. PROOF Write a two-column proof. (Lesson 5-3)

Given: $\overline{RQ}$ bisects $\angle SRT$.

Prove: $m\angle SQR > m\angle SRQ$

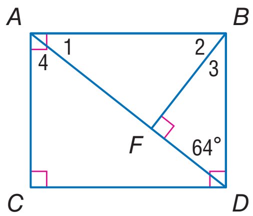


COORDINATE GEOMETRY **Find the coordinates of the circumcenter of each triangle with the given vertices.** (Lesson 5-1)

50. $D(-3, 3)$, $E(3, 2)$, $F(1, -4)$

51. $A(4, 0)$, $B(-2, 4)$, $C(0, 6)$

Find each measure. (Lesson 4-2)



52. $m\angle 1$

53. $m\angle 4$

COORDINATE GEOMETRY **Find the distance between each pair of parallel lines with the given equations.** (Lesson 3-6)

54. $x + 3y = 6$
$x + 3y = -14$

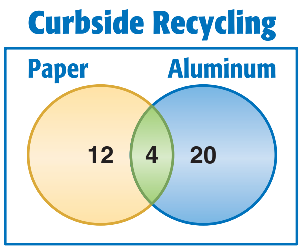
55. $y = 2x + 2$
$y = 2x - 3$

56. RECYCLING Refer to the Venn diagram that represents the number of neighborhoods in a city with a curbside recycling program for paper or aluminum. (Lesson 2-2)

a. How many neighborhoods recycle aluminum?

b. How many neighborhoods recycle paper or aluminum or both?

c. How many neighborhoods recycle paper and aluminum?



Skills Review

Determine whether each inequality is *true* or *false*.

57. $23 - 11 > 9$

58. $41 - 19 < 21$

59. $57 + 68 < 115$