LESSON 5-6

Inequalities in Two Triangles

Then

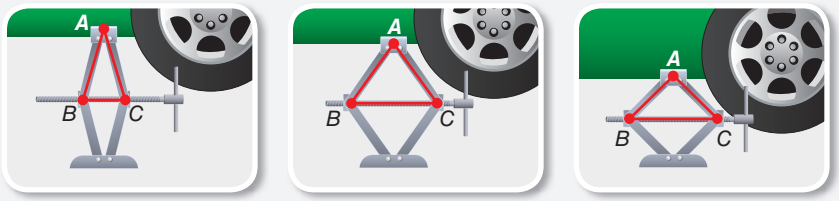
- You used inequalities to make comparisons in one triangle.

Now

1. Apply the Hinge Theorem or its converse to make comparisons in two triangles.
2. Prove triangle relationships using the Hinge Theorem or its converse.

Why?

- A car jack is used to lift a car. The jack shown below is one of the simplest still in use today. Notice that as the jack is lowered, the legs of isosceles $\triangle ABC$ remain congruent, but the included angle A widens and $\overline{BC}$, the side opposite $\angle A$, lengthens.



Common Core State Standards

Content Standards

G.CO.10 Prove theorems about triangles.

Mathematical Practices

3 Construct viable arguments and critique the reasoning of others.

1 Make sense of problems and persevere in solving them.

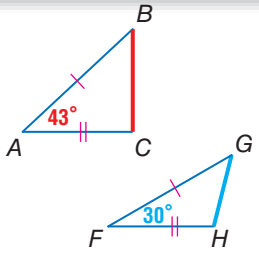
1 Hinge Theorem

The observation in the example above is true of any type of triangle and illustrates the following theorems.

Theorems Inequalities in Two Triangles

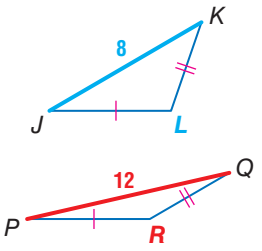
5.13 Hinge Theorem If two sides of a triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second triangle, then the third side of the first triangle is longer than the third side of the second triangle.

Example: If $\overline{AB} \cong \overline{FG}$, $\overline{AC} \cong \overline{FH}$, and $m\angle A > m\angle F$, then $BC > GH$.



5.14 Converse of the Hinge Theorem If two sides of a triangle are congruent to two sides of another triangle, and the third side in the first is longer than the third side in the second triangle, then the included angle measure of the first triangle is greater than the included angle measure in the second triangle.

Example: If $\overline{JL} \cong \overline{PR}$, $\overline{KL} \cong \overline{QR}$, and $PQ > JK$, then $m\angle R > m\angle L$.

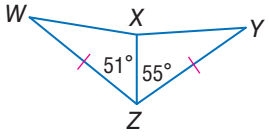


The proof of Theorem 5.13 is on p. 372. You will prove Theorem 5.14 in Exercise 28.

Example 1 Use the Hinge Theorem and its Converse

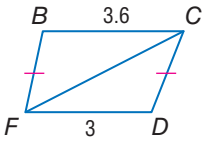
Compare the given measures.

a. WX and XY



In $\triangle WXZ$ and $\triangle YXZ$, $\overline{WZ} \cong \overline{YZ}$, $\overline{XZ} \cong \overline{XZ}$, and $\angle YZX > \angle WZX$. By the Hinge Theorem, $m\angle WZX < m\angle YZX$, so $WX < XY$.

b. $m\angle FCD$ and $m\angle BFC$

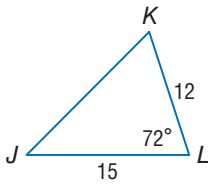


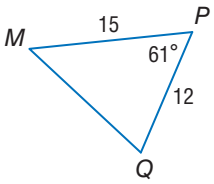
In $\triangle BCF$ and $\triangle DFC$, $\overline{BF} \cong \overline{DC}$, $\overline{FC} \cong \overline{CF}$, and $BC > FD$. By the Converse of the Hinge Theorem, $\angle BFC > \angle DCF$.

GuidedPractice

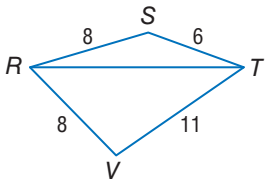
Compare the given measures.

1A. JK and MQ





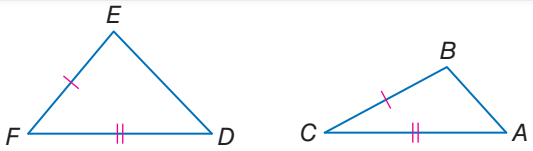
1B. $m\angle SRT$ and $m\angle VRT$



Proof Hinge Theorem

Given: $\triangle ABC$ and $\triangle DEF$, $\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$, $m\angle F > m\angle C$

Prove: $DE > AB$



Proof:

We are given that $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$. We also know that $m\angle F > m\angle C$.

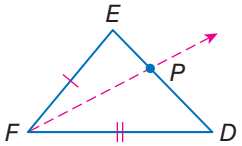
Draw auxiliary ray FP such that $m\angle DFP = m\angle C$ and that $\overline{PF} \cong \overline{BC}$. This leads to two cases.

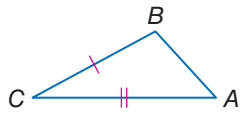
StudyTip

SAS and SSS Inequality Theorem The Hinge Theorem is also called the SAS Inequality Theorem. The Converse of the Hinge Theorem is also called the SSS Inequality Theorem.

Case 1 P lies on $\overline{DE}$.

Then $\triangle FPD \cong \triangle CBA$ by SAS. Thus, $PD = BA$ by CPCTC and the definition of congruent segments.

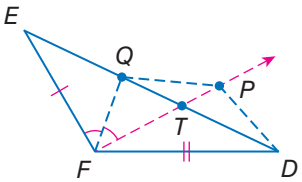


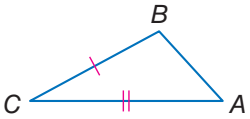


By the Segment Addition Postulate, $DE = EP + PD$. Also, $DE > PD$ by the definition of inequality. Therefore, $DE > AB$ by substitution.

Case 2 P does not lie on $\overline{DE}$.

Then let the intersection of $\overline{FP}$ and $\overline{ED}$ be point T, and draw another auxiliary segment $\overline{FQ}$ such that Q is on $\overline{DE}$ and $\angle EFQ \cong \angle QFP$. Then draw auxiliary segments $\overline{PD}$ and $\overline{PQ}$.





Since $\overline{FP} \cong \overline{BC}$ and $\overline{BC} \cong \overline{EF}$, we have $\overline{FP} \cong \overline{EF}$ by the Transitive Property. Also $\overline{QF}$ is congruent to itself by the Reflexive Property. Thus, $\triangle EFQ \cong \triangle PFQ$ by SAS. By CPCTC, $\overline{EQ} \cong \overline{PQ}$ or $EQ = PQ$. Also, $\triangle FPD \cong \triangle CBA$ by SAS. So, $\overline{PD} \cong \overline{BA}$ by CPCTC and $PD = BA$.

In $\triangle QPD$, $QD + PQ > PD$ by the Triangle Inequality Theorem. By substitution, $QD + EQ >$ PD. Since $ED = QD + EQ$ by the Segment Addition Postulate, $ED > PD$. Using substitution, $ED > BA$ or $DE > AB$.

Real-World Link

There are over 225,000 miles of groomed and marked snowmobile trails in North America.

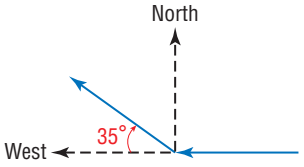
Source: International Snowmobile Manufacturers Association

You can use the Hinge Theorem to solve real-world problems.

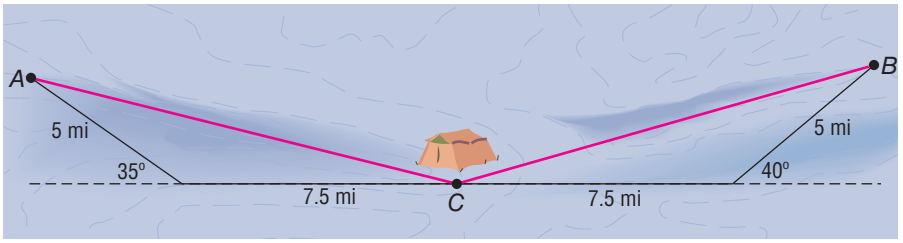
Real-World Example 2 Use the Hinge Theorem

SNOWMOBILING **Two groups of snowmobilers leave from the same base camp. Group A goes 7.5 miles due west and then turns 35° north of west and goes 5 miles. Group B goes 7.5 miles due east and then turns 40° north of east and goes 5 miles. At this point, which group is farther from the base camp? Explain your reasoning.**

Understand Using the sets of directions given in the problem, you need to determine which snowmobile group is farther from the base camp. A turn of 35° north of west is correctly interpreted as shown.



Plan Draw a diagram of the situation.



Problem-Solving Tip

Draw a Diagram Draw a diagram to help you see and correctly interpret a problem that has been described in words.

The paths taken by each group and the straight-line distance back to the camp form two triangles. Each group goes 7.5 miles and then turns and goes 5 miles.

Use linear pairs to find the measures of the included angles. Then apply the Hinge Theorem to compare the distance each group is from base camp.

Solve The included angle for the path made by Group A measures $180 - 35$ or 145. The included angle for the path made by Group B is $180 - 40$ or 140.

Since $145 > 140$, $AC > BC$ by the Hinge Theorem. So Group A is farther from the base camp.

Check Group B turned 5° more than Group A did back toward base camp, so they should be closer to base camp than Group A. Thus, Group A should be farther from the base camp. ✔

Guided Practice

2A. SKIING Two groups of skiers leave from the same lodge. Group A goes 4 miles due east and then turns 70° north of east and goes 3 miles. Group B goes 4 miles due west and then turns 75° north of west and goes 3 miles. At this point, which group is *farther* from the lodge? Explain your reasoning.

2B. SKIING In problem 2A, suppose Group A instead went 4 miles west and then turned 45° north of west and traveled 3 miles. Which group would be *closer* to the lodge? Explain your reasoning.

When the included angle of one triangle is greater than the included angle in a second triangle, the Converse of the Hinge Theorem is used.

StudyTip

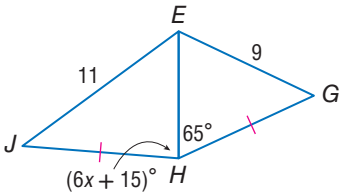
Using Additional Facts

When finding a range for the possible values for x, you may need to use one of the following facts.

- The measure of any angle is always greater than 0 and less than 180.
- The measure of any segment is always greater than 0.

Example 3 Apply Algebra to the Relationships in Triangles

ALGEBRA **Find the range of possible values for x.**



Step 1 From the diagram, we know that $\overline{JH} \cong \overline{GH}$, $\overline{EH} \cong \overline{EH}$, and $JE > EG$.

$m\angle JHE > m\angle EHG$ Converse of the Hinge Theorem

$6x + 15 > 65$ Substitution

$x > 8\frac{1}{3}$ Solve for x.

Step 2 Use the fact that the measure of any angle in a triangle is less than 180 to write a second inequality.

$m\angle JHE < 180$

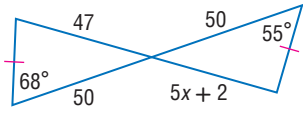
$6x + 15 < 180$ Substitution

$x < 27.5$ Solve for x.

Step 3 Write $x > 8\frac{1}{3}$ and $x < 27.5$ as the compound inequality $8\frac{1}{3} < x < 27.5$.

GuidedPractice

3. Find the range of possible values for x.



2 Prove Relationships In Two Triangles

You can use the Hinge Theorem and its converse to prove relationships in two triangles.

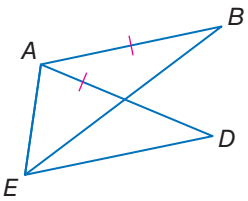
Example 4 Prove Triangle Relationships Using Hinge Theorem

Write a two-column proof.

Given: $\overline{AB} \cong \overline{AD}$

Prove: $EB > ED$

Proof:



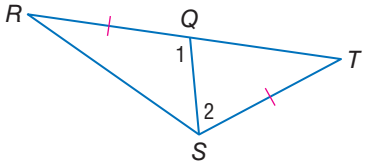
Statements	Reasons
1. $\overline{AB} \cong \overline{AD}$	1. Given
2. $\overline{AE} \cong \overline{AE}$	2. Reflexive Property
3. $m\angle EAB = m\angle EAD + m\angle DAB$	3. Angle Addition Postulate
4. $m\angle EAB > m\angle EAD$	4. Definition of Inequality
5. $EB > ED$	5. Hinge Theorem

GuidedPractice

4. Write a two-column proof.

Given: $\overline{RQ} \cong \overline{ST}$

Prove: $RS > TQ$

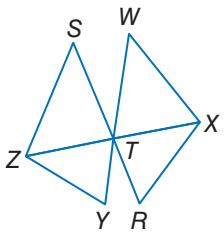


Example 5 Prove Relationships Using Converse of Hinge Theorem

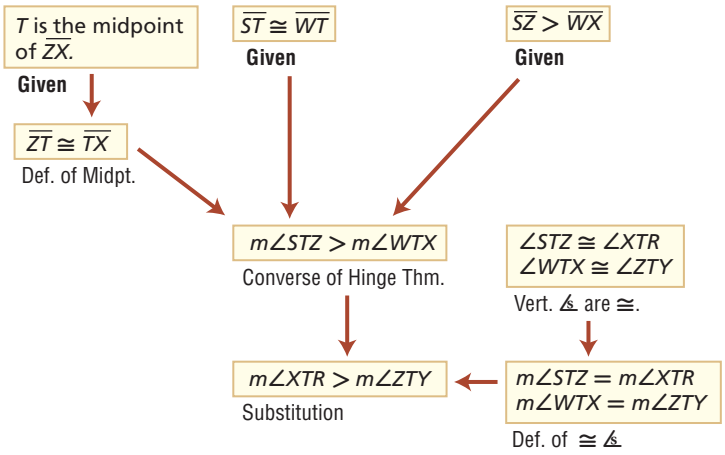
Write a flow proof.

Given: T is the midpoint of $\overline{ZX}$.
$\overline{ST} \cong \overline{WT}$
$SZ > WX$

Prove: $m\angle XTR > m\angle ZTY$



Flow Proof:

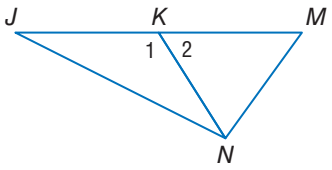


GuidedPractice

5. Write a two-column proof.

Given: $\overline{NK}$ is a median of $\triangle JMN$.
$JN > NM$

Prove: $m\angle 1 > m\angle 2$

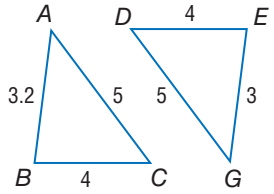


Check Your Understanding

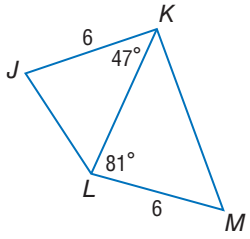
= Step-by-Step Solutions begin on page R14.

Example 1 **Compare the given measures.**

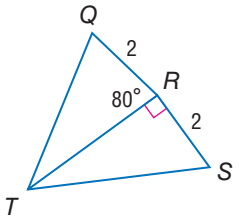
1. $m\angle ACB$ and $m\angle GDE$



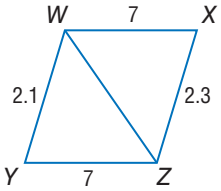
2. JL and KM



3 QT and ST



4. $m\angle XWZ$ and $m\angle YZW$

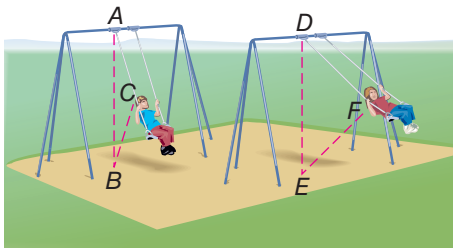


Example 2

5. SWINGS The position of the swing changes based on how hard the swing is pushed.

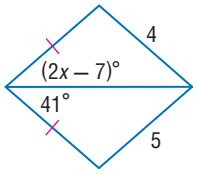
a. Which pairs of segments are congruent?

b. Is the measure of $\angle A$ or the measure of $\angle D$ greater? Explain.



Example 3 **Find the range of possible values for x.**

6.



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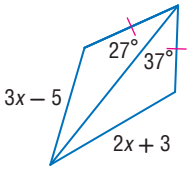
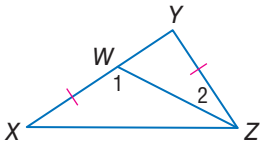
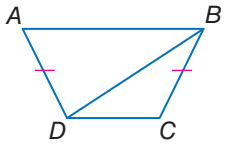


Examples 4–5 CCSS **ARGUMENTS** **Write a two-column proof.**

8. Given: $\triangle YZX$
$\overline{YZ} \cong \overline{XW}$
Prove: $ZX > YW$



9. Given: $\overline{AD} \cong \overline{CB}$
$DC < AB$
Prove: $m\angle CBD < m\angle ADB$

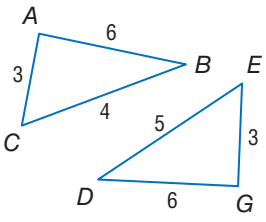


Practice and Problem Solving

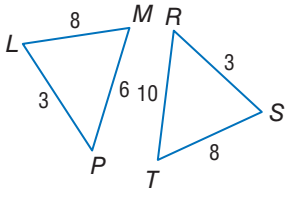
Extra Practice is on page R5.

Example 1 **Compare the given measures.**

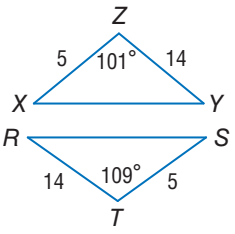
10. $m\angle BAC$ and $m\angle DGE$



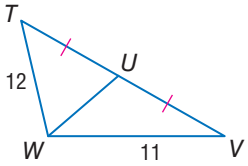
11. $m\angle MLP$ and $m\angle TSR$



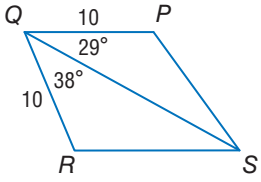
12. SR and XY



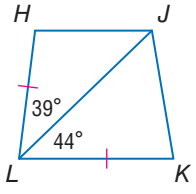
13. $m\angle TUW$ and $m\angle VUW$



14. PS and SR



15. JK and HJ



Example 2

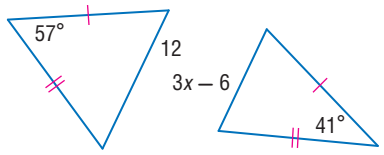
16. CAMPING Pedro and Joel are camping in a national park. One morning, Pedro decides to hike to the waterfall. He leaves camp and goes 5 miles east then turns 15° south of east and goes 2 more miles. Joel leaves the camp and travels 5 miles west, then turns 35° north of west and goes 2 miles to the lake for a swim.

a. When they reach their destinations, who is closer to the camp? Explain your reasoning. Include a diagram.

b. Suppose instead of turning 35° north of west, Joel turned 10° south of west. Who would then be farther from the camp? Explain your reasoning. Include a diagram.

Example 3 Find the range of possible values for x.

17.



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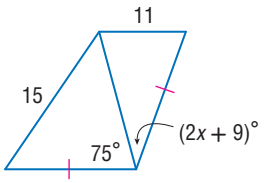

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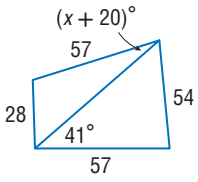

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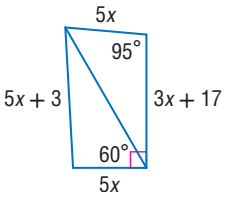
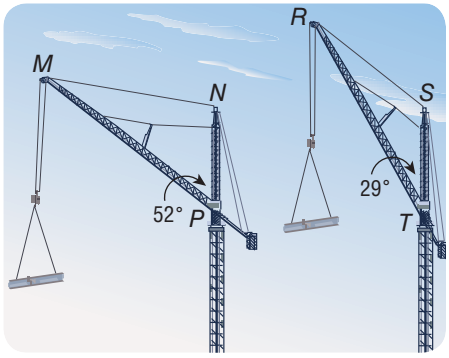
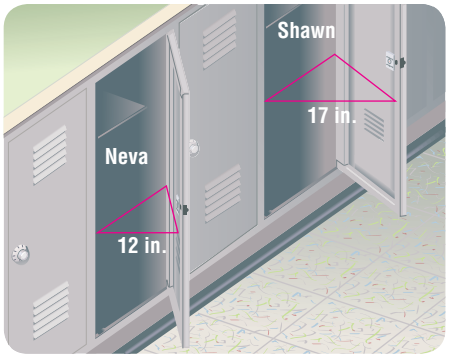


21. CRANES In the diagram, a crane is shown lifting an object to two different heights. The length of the crane's arm is fixed, and $\overline{MP} \cong \overline{RT}$. Is $\overline{MN}$ or $\overline{RS}$ shorter? Explain your reasoning.



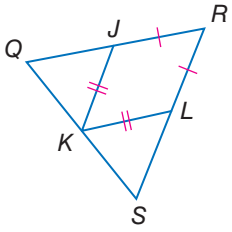
22. LOCKERS Neva and Shawn both have their lockers open as shown in the diagram. Whose locker forms a larger angle? Explain your reasoning.



Examples 4–5 CCSS **ARGUMENTS** Write a two-column proof.

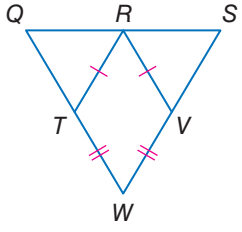
23. **Given:** $\overline{LK} \cong \overline{JK}$, $\overline{RL} \cong \overline{RJ}$
K is the midpoint of $\overline{QS}$.
$m\angle SKL > m\angle QKJ$

Prove: $RS > QR$



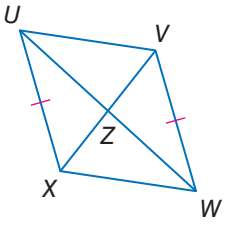
24. **Given:** $\overline{VR} \cong \overline{RT}$, $\overline{WV} \cong \overline{WT}$
$m\angle SRV > m\angle QRT$
R is the midpoint of $\overline{SQ}$.

Prove: $WS > WQ$



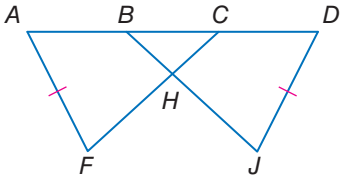
25. **Given:** $\overline{XU} \cong \overline{VW}$, $VW > XW$
$\overline{XU} \parallel \overline{VW}$

Prove: $m\angle XZU > m\angle UZV$

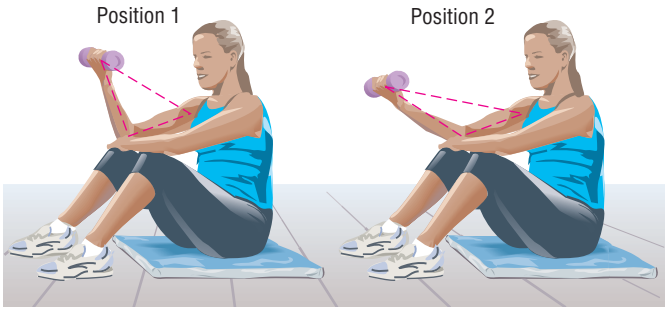


26. **Given:** $\overline{AF} \cong \overline{DJ}$, $\overline{FC} \cong \overline{JB}$
$AB > DC$

Prove: $m\angle AFC > m\angle DJB$



27 **EXERCISE** Anica is doing knee-supported bicep curls as part of her strength training.



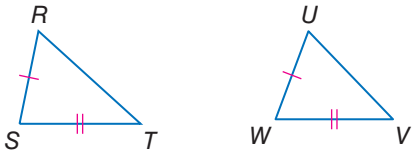
a. Is the distance from Anica's fist to her shoulder greater in Position 1 or Position 2? Justify your answer using measurement.

b. Is the measure of the angle formed by Anica's elbow greater in Position 1 or Position 2? Explain your reasoning.

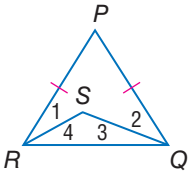
28. **PROOF** Use an indirect proof to prove the SSS Inequality Theorem (Theorem 5.14).

Given: $\overline{RS} \cong \overline{UW}$
$\overline{ST} \cong \overline{WV}$
$RT > UV$

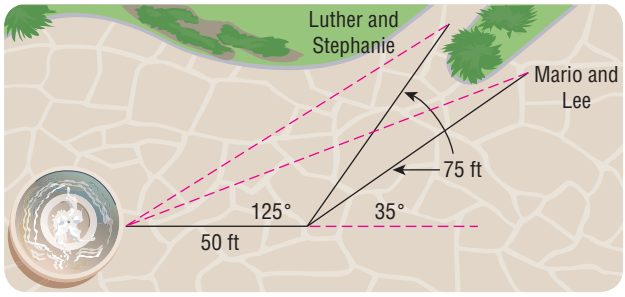
Prove: $m\angle S > m\angle W$



29. **PROOF** If $\overline{PR} \cong \overline{PQ}$ and $SQ > SR$, write a two-column proof to prove $m\angle 1 < m\angle 2$.



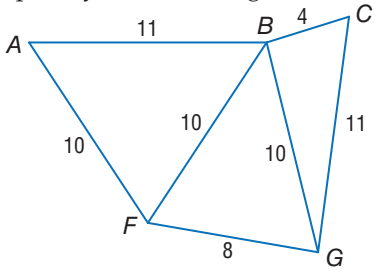
30. **SCAVENGER HUNT** Stephanie, Mario, Lee, and Luther are participating in a scavenger hunt as part of a geography lesson. Their map shows that the next clue is 50 feet due east and then 75 feet 35° east of north starting from the fountain in the school courtyard. When they get ready to turn and go 75 feet 35° east of north, they disagree about which way to go, so they split up and take the paths shown in the diagram below.



a. Which pair chose the correct path? Explain your reasoning.

b. Which pair is closest to the fountain when they stop? Explain your reasoning.

CCSS **SENSE-MAKING** **Use the figure at the right to write an inequality relating the given pair of angle or segment measures.**

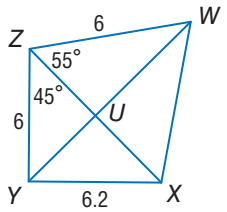


31. CB and AB

32. $m\angle FBG$ and $m\angle BFA$

33. $m\angle BGC$ and $m\angle FBA$

Use the figure at the right to write an inequality relating the given pair of angles or segment measures.



34. $m\angle ZUY$ and $m\angle ZUW$

35 WU and YU

36. WX and XY

37. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate properties of polygons.

a. Geometric Draw a three-sided, a four-sided, and a five-sided polygon. Label the 3-sided polygon ABC, the four-sided polygon $FGHJ$, and the five-sided polygon $PQRST$. Use a protractor to measure and label each angle.

b. Tabular Copy and complete the table below.

Number of sides	Angle Measures				Sum of Angles
3	$m\angle A$		$m\angle C$		
	$m\angle B$				
4	$m\angle F$		$m\angle H$		
	$m\angle G$		$m\angle J$		
5	$m\angle P$		$m\angle S$		
	$m\angle Q$		$m\angle T$		
	$m\angle R$				

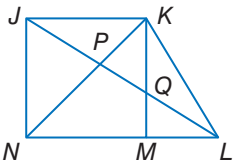
c. Verbal Make a conjecture about the relationship between the number of sides of a polygon and the sum of the measures of the angles of the polygon.

d. Logical What type of reasoning did you use in part **c**? Explain.

e. Algebraic Write an algebraic expression for the sum of the measures of the angles for a polygon with n sides.

H.O.T. Problems Use Higher-Order Thinking Skills

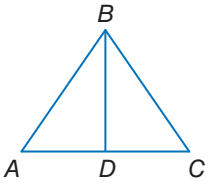
38. **CHALLENGE** If $m\angle LJN > m\angle KJL$, $KJ \cong JN$, and $JN \perp NL$, which is greater, $m\angle LKN$ or $m\angle LNK$? Explain your reasoning.



39. **OPEN ENDED** Give a real-world example of an object that uses a hinge. Draw two sketches in which the hinge on your object is adjusted to two different positions. Use your sketches to explain why Theorem 5.13 is called the Hinge Theorem.

40. **CHALLENGE** Given $\triangle RST$ with median $\overline{RQ}$, if RT is greater than or equal to RS, what are the possible classifications of $\triangle RQT$? Explain your reasoning.

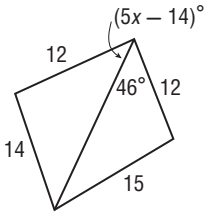
41. **CCSS PRECISION** If $\overline{BD}$ is a median and $AB < BC$, then $\angle BDC$ is *always*, *sometimes*, or *never* an acute angle. Explain.



42. **WRITING IN MATH** Compare and contrast the Hinge Theorem to the SAS Postulate for triangle congruence.

Standardized Test Practice

43. SHORT RESPONSE Write an inequality to describe the possible range of values for x.



44. Which of the following is the inverse of the statement *If it is snowing, then Steve wears his snow boots?*

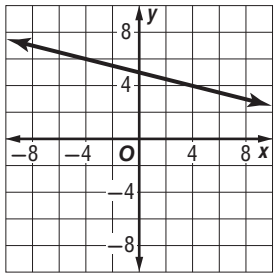
A If Steve wears his snow boots, then it is snowing.

B If it is not snowing, then Steve does not wear his snow boots.

C If it is not snowing, then Steve wears his snow boots.

D If it never snows, then Steve does not own snow boots.

45. ALGEBRA Which linear function best describes the graph shown?



F $y = -\frac{1}{4}x + 5$

G $y = -\frac{1}{4}x - 5$

H $y = \frac{1}{4}x + 5$

J $y = \frac{1}{4}x - 5$

46. SAT/ACT If the side of a square is $x + 3$, then the diagonal of the square is

A $x^2 + 1$

B $x\sqrt{2} + 3\sqrt{2}$

C $2x + 6$

D $x^2\sqrt{2} + 6$

E $x^2 + 9$

Spiral Review

Find the range for the measure of the third side of a triangle given the measures of two sides. (Lesson 5-5)

47. 3.2 cm, 4.4 cm

48. 5 ft, 10 ft

49. 3 m, 9 m

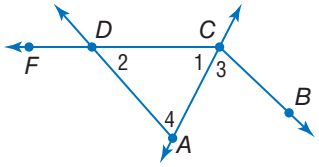
50. CRUISES Ally asked Tavia the cost of a cruise she and her best friend went on after graduation. Tavia could not remember how much it cost per person, but she did remember that the total cost was over $500. Use indirect reasoning to show that the cost for one person was more than $250. (Lesson 5-4)

Draw and label a figure to represent the congruent triangles. Then find x. (Lesson 4-3)

51. $\triangle QRS \cong \triangle GHJ$, $RS = 12$, $QR = 10$, $QS = 6$, and $HJ = 2x - 4$.

52. $\triangle ABC \cong \triangle XYZ$, $AB = 13$, $AC = 19$, $BC = 21$, and $XY = 3x + 7$.

Use the figure at the right. (Lesson 1-4)



53. Name the vertex of $\angle 4$.

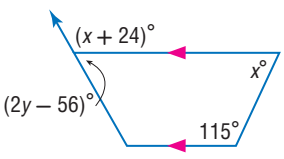
54. What is another name for $\angle 2$?

55. What is another name for $\angle BCA$?

Skills Review

Find the value of the variable(s) in each figure. Explain your reasoning.

56.



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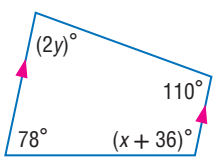

58.