LESSON 8-1 Geometric Mean

Then

- You used proportional relationships of corresponding angle bisectors, altitudes, and medians of similar triangles.

Now

- 1 Find the geometric mean between two numbers.
- 2 Solve problems involving relationships between parts of a right triangle and the altitude to its hypotenuse.

Why?

- Photographing very tall or very wide objects can be challenging. It can be difficult to include the entire object in your shot without distorting the image. If your camera is set for a vertical viewing angle of 90° and you know the height of the object you wish to photograph, you can use the geometric mean of the distance from the top of the object to your camera level and the distance from the bottom of the object to camera level.

NewVocabulary

geometric mean

Common Core State Standards

Content Standards

G.SRT.4 Prove theorems about triangles.

G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Mathematical Practices

7 Look for and make use of structure.

3 Construct viable arguments and critique the reasoning of others.

1 Geometric Mean

When the means of a proportion are the same number, that number is called the geometric mean of the extremes. The **geometric mean** between two numbers is the positive square root of their product.

$$\text{extreme} \rightarrow \frac{a}{x} \leftarrow \text{mean} \qquad = \qquad \text{mean} \rightarrow \frac{x}{b} \leftarrow \text{extreme}$$

KeyConcept Geometric Mean

Words	The geometric mean of two positive numbers a and b is the number x such that $\frac{a}{x} = \frac{x}{b}$. So, $x^2 = ab$ and $x = \sqrt{ab}$.
Example	The geometric mean of $a = 9$ and $b = 4$ is 6, because $6 = \sqrt{9 \cdot 4}$.

Example 1 Geometric Mean

PT

Find the geometric mean between 8 and 10.

$x = \sqrt{ab}$	Definition of geometric mean
$= \sqrt{8 \cdot 10}$	$a = 8$ and $b = 10$
$= \sqrt{(4 \cdot 2) \cdot (2 \cdot 5)}$	Factor.
$= \sqrt{16 \cdot 5}$	Associative Property
$= 4\sqrt{5}$	Simplify.

The geometric mean between 8 and 10 is $4\sqrt{5}$ or about 8.9.

GuidedPractice

Find the geometric mean between each pair of numbers.

1A. 5 and 45

1B. 12 and 15

2 Geometric Means in Right Triangles

In a right triangle, an altitude drawn from the vertex of the right angle to the hypotenuse forms two additional right triangles. These three right triangles share a special relationship.

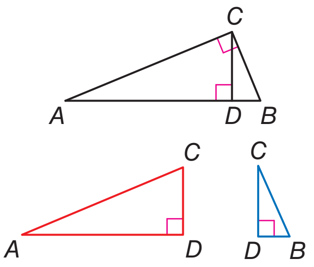
Review Vocabulary

altitude (of a triangle) a segment from a vertex to the line containing the opposite side and perpendicular to the line containing that side

Theorem 8.1

If the altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.

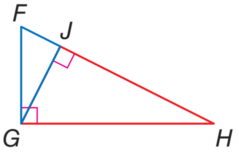
Example If $\overline{CD}$ is the altitude to hypotenuse $\overline{AB}$ of right $\triangle ABC$, then $\triangle ACD \sim \triangle ABC$, $\triangle CBD \sim \triangle ABC$, and $\triangle ACD \sim \triangle CBD$.



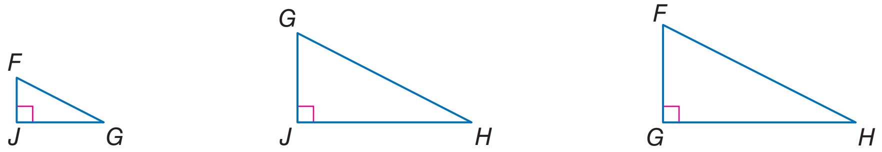
You will prove Theorem 8.1 in Exercise 39.

Example 2 Identify Similar Right Triangles

Write a similarity statement identifying the three similar right triangles in the figure.



Separate the triangle into two triangles along the altitude. Then sketch the three triangles, reorienting the smaller ones so that their corresponding angles and sides are in the same positions as the original triangle.



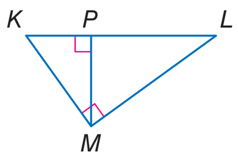
So by Theorem 8.1, $\triangle FJG \sim \triangle GJH \sim \triangle FGH$.

Study Tip

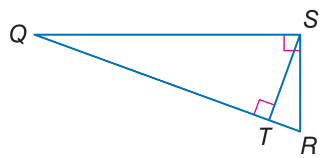
Reorienting Triangles To reorient the right triangles in Example 2, first match up the right angles. Then match up the shorter sides.

Guided Practice

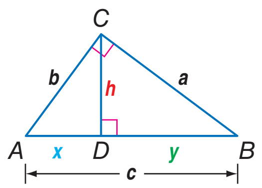
2A.

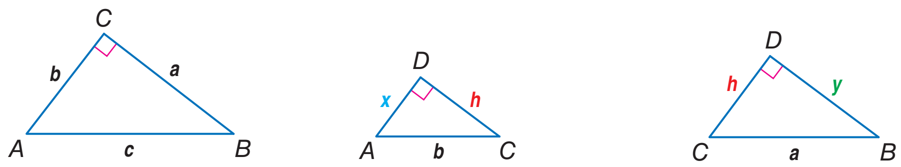


2B.



From Theorem 8.1, you know that altitude $\overline{CD}$ drawn to the hypotenuse of right triangle ABC forms three similar triangles: $\triangle ACB \sim \triangle ADC \sim \triangle CDB$. By the definition of similar polygons, you can write the following proportions comparing the side lengths of these triangles.



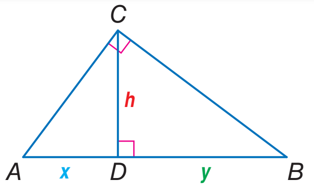


$$\frac{\text{shorter leg}}{\text{longer leg}} = \frac{b}{a} = \frac{x}{h} = \frac{h}{y} \qquad \frac{\text{hypotenuse}}{\text{shorter leg}} = \frac{c}{b} = \frac{b}{x} = \frac{a}{h} \qquad \frac{\text{hypotenuse}}{\text{longer leg}} = \frac{c}{a} = \frac{b}{h} = \frac{a}{y}$$

Notice that the circled relationships involve geometric means. This leads to the theorems at the top of the next page.

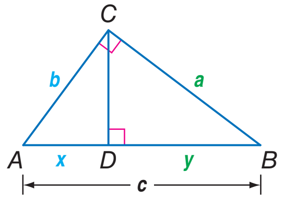
Theorems Right Triangle Geometric Mean Theorems

8.2 Geometric Mean (Altitude) Theorem The altitude drawn to the hypotenuse of a right triangle separates the hypotenuse into two segments. The length of this altitude is the geometric mean between the lengths of these two segments.



Example If $\overline{CD}$ is the altitude to hypotenuse $\overline{AB}$ of right $\triangle ABC$, then $\frac{x}{h} = \frac{h}{y}$ or $h = \sqrt{xy}$.

8.3 Geometric Mean (Leg) Theorem The altitude drawn to the hypotenuse of a right triangle separates the hypotenuse into two segments. The length of a leg of this triangle is the geometric mean between the length of the hypotenuse and the segment of the hypotenuse adjacent to that leg.

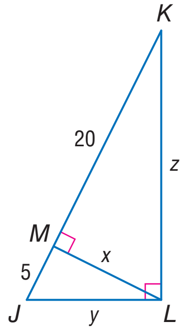


Example If $\overline{CD}$ is the altitude to hypotenuse $\overline{AB}$ of right $\triangle ABC$, then $\frac{c}{b} = \frac{b}{x}$ or $b = \sqrt{xc}$ and $\frac{c}{a} = \frac{a}{y}$ or $a = \sqrt{yc}$.

You will prove Theorems 8.2 and 8.3 in Exercises 40 and 41, respectively.

Example 3 Use Geometric Mean with Right Triangles

Find *x*, *y*, and *z*.



Since x is the measure of the altitude drawn to the hypotenuse of right $\triangle JKL$, x is the geometric mean of the lengths of the two segments that make up the hypotenuse, JM and MK.

$x = \sqrt{JM \cdot MK}$ Geometric Mean (Altitude) Theorem

$= \sqrt{5 \cdot 20}$ Substitution

$= \sqrt{100}$ or 10 Simplify.

Since y is the measure of leg $\overline{JL}$, y is the geometric mean of $\overline{JM}$, the measure of the segment adjacent to this leg, and the measure of the hypotenuse JK.

$y = \sqrt{JM \cdot JK}$ Geometric Mean (Leg) Theorem

$= \sqrt{5 \cdot (20 + 5)}$ Substitution

$= \sqrt{125}$ or about 11.2 Use a calculator to simplify.

Since z is the measure of leg $\overline{KL}$, z is the geometric mean of $\overline{MK}$, the measure of the segment adjacent to $\overline{KL}$, and the measure of the hypotenuse JK.

$z = \sqrt{MK \cdot JK}$ Geometric Mean (Leg) Theorem

$= \sqrt{20 \cdot (20 + 5)}$ Substitution

$= \sqrt{500}$ or about 22.4 Use a calculator to simplify.

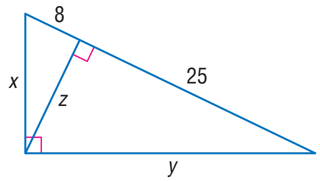
StudyTip

Use a Proportion In Example 3, the value of x could also be found by solving the proportion $\frac{5}{x} = \frac{x}{20}$.

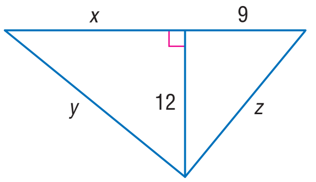
GuidedPractice

Find x, y, and z.

3A.



3B.



You can use geometric mean to measure height indirectly.

Real-WorldCareer

Event Planner

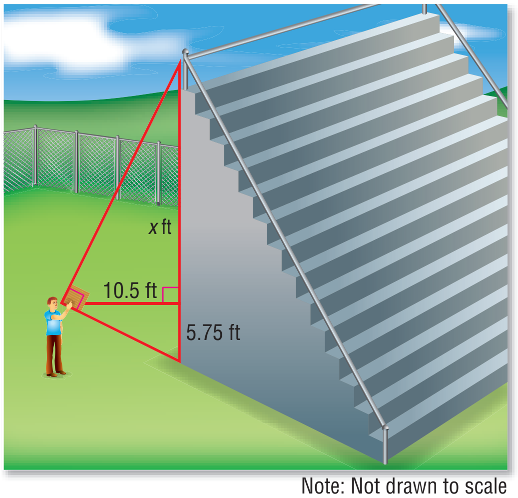
Event planners organize events including choosing a location, arranging for food, and scheduling entertainment. They also coordinate services like transportation and photography.

Most of the skills required for event planning are acquired through on-the-job experience.

Real-World Example 4 Indirect Measurement

ADVERTISING Zach wants to order a banner that will hang over the side of his high school baseball stadium grandstand and reach the ground.

To find this height, he uses a cardboard square to line up the top and bottom of the grandstand. He measures his distance from the grandstand and from the ground to his eye level. Find the height of the grandstand to the nearest foot.



Note: Not drawn to scale

The distance from Zach to the grandstand is the altitude to the hypotenuse of a right triangle. The length of this altitude is the geometric mean of the two segments that make up the hypotenuse. The shorter segment has the measure of 5.75 feet. Let the unknown measure be x feet.

$10.5 = \sqrt{5.75 \cdot x}$	Geometric Mean (Altitude) Theorem
$110.25 = 5.75x$	Square each side.
$19.17 \approx x$	Divide each side by 5.75.

The height of the grandstand is the total length of the hypotenuse, $5.75 + 19.17$, or about 25 feet.

GuidedPractice

4. **SPORTS** A community center needs to estimate the cost of installing a rock climbing wall by estimating the height of the wall. Sue holds a book up to her eyes so that the top and bottom of the wall are in line with the bottom edge and binding of the cover. If her eye level is 5 feet above the ground and she stands 11 feet from the wall, how high is the wall? Draw a diagram and explain your reasoning.

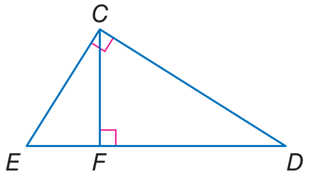
Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Example 1 **Find the geometric mean between each pair of numbers.**

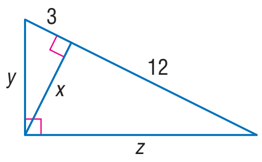
1. 5 and 20 **2.** 36 and 4 **3.** 40 and 15

Example 2 **4.** Write a similarity statement identifying the three similar triangles in the figure.



Example 3 **Find *x*, *y*, and *z*.**

5.



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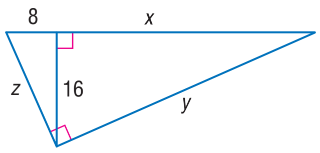
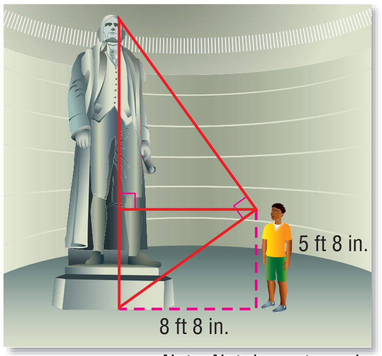


Example 4 **7.** CCSS **MODELING** Corey is visiting the Jefferson Memorial with his family. He wants to estimate the height of the statue of Thomas Jefferson. Corey stands so that his line of vision to the top and base of the statue form a right angle as shown in the diagram. About how tall is the statue?



Note: Not drawn to scale.

Practice and Problem Solving

Extra Practice is on page R8.

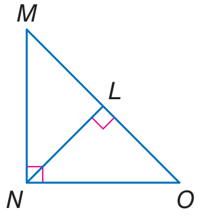
Example 1 **Find the geometric mean between each pair of numbers.**

8. 81 and 4 **9.** 25 and 16 **10.** 20 and 25

11. 36 and 24 **12.** 12 and 2.4 **13.** 18 and 1.5

Example 2 **Write a similarity statement identifying the three similar triangles in the figure.**

14.



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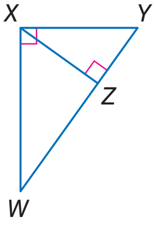

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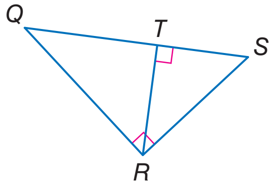

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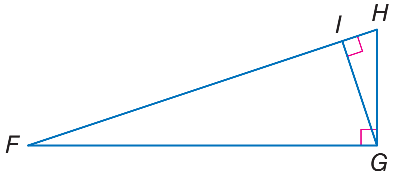
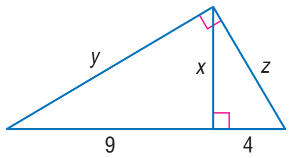


Example 3 **Find x, y, and z.**

18.



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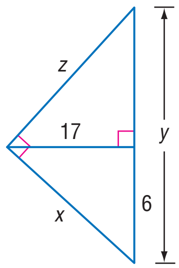

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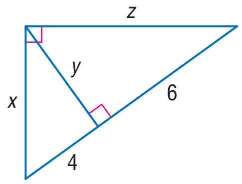

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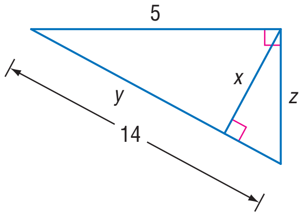

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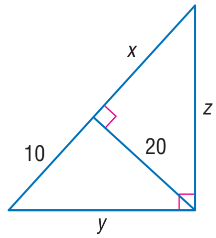

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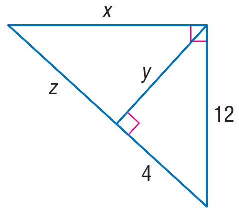
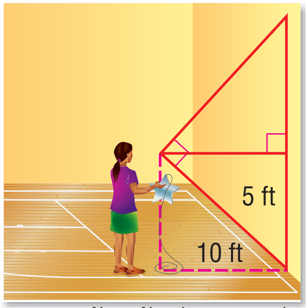
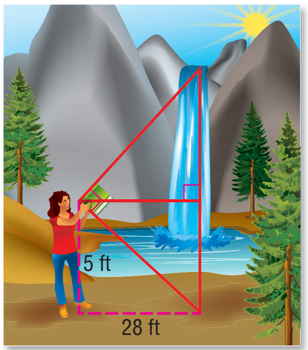


Example 4 **24.** **CCSS MODELING** Evelina is hanging silver stars from the gym ceiling using string for the homecoming dance. She wants the ends of the strings where the stars will be attached to be 7 feet from the floor. Use the diagram to determine how long she should make the strings.



Note: Not drawn to scale.

25. **CCSS MODELING** Makayla is using a book to sight the top of a waterfall. Her eye level is 5 feet from the ground and she is a horizontal distance of 28 feet from the waterfall. Find the height of the waterfall to the nearest tenth of a foot.



Note: Not drawn to scale

Find the geometric mean between each pair of numbers.

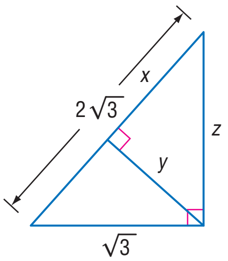
26. $\frac{1}{5}$ and 60

27. $\frac{3\sqrt{2}}{7}$ and $\frac{5\sqrt{2}}{7}$

28. $\frac{3\sqrt{5}}{4}$ and $\frac{5\sqrt{5}}{4}$

Find x, y, and z.

29.



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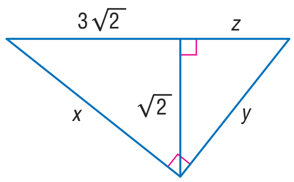
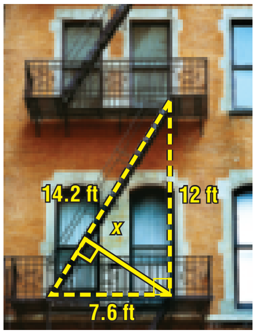


31. **ALGEBRA** The geometric mean of a number and four times the number is 22. What is the number?

Use similar triangles to find the value of x.

32.



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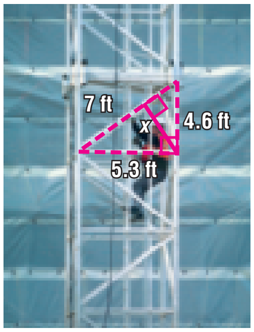

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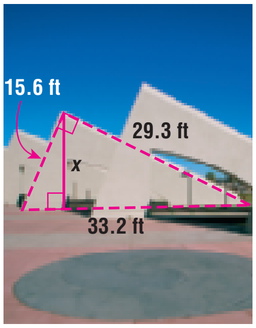
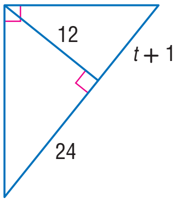


ALGEBRA **Find the value of the variable.**

35.



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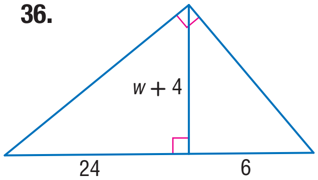

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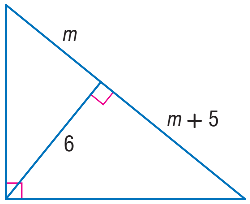
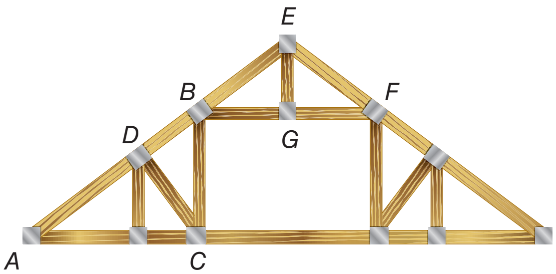


38. CONSTRUCTION A room-in-attic truss is a truss design that provides support while leaving area that can be enclosed as living space. In the diagram, $\angle BCA$ and $\angle EGB$ are right angles, $\triangle BEF$ is isosceles, $\overline{CD}$ is an altitude of $\triangle ABC$, and $\overline{EG}$ is an altitude of $\triangle BEF$. If $DB = 5$ feet, $CD = 6$ feet 4 inches, $BF = 10$ feet 10 inches, and $EG = 4$ feet 6 inches, what is AE?



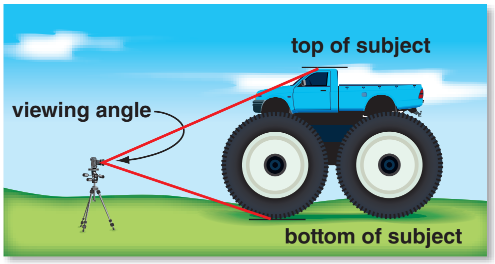
CCSS **ARGUMENTS** **Write a proof for each theorem.**

39. Theorem 8.1

40. Theorem 8.2

41. Theorem 8.3

42. TRUCKS In photography, the angle formed by the top of the subject, the camera, and the bottom of the subject is called the viewing angle, as shown at the right. Natalie is taking a picture of Bigfoot #5, which is 15 feet 6 inches tall. She sets her camera on a tripod that is 5 feet above ground level. The vertical viewing angle of her camera is set for 90°.



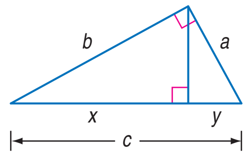
a. Sketch a diagram of this situation.

b. How far away from the truck should Natalie stand so that she perfectly frames the entire height of the truck in her shot?

43 **FINANCE** The average rate of return on an investment over two years is the geometric mean of the two annual returns. If an investment returns 12% one year and 7% the next year, what is the average rate of return on this investment over the two-year period?

(tl)BananaStock/Punchstock, (tc)Image Source/SuperStock, (tr)Richard Cummins/SuperStock

44. **PROOF** Derive the Pythagorean Theorem using the figure at the right and the Geometric Mean (Leg) Theorem.



Determine whether each statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

45 The geometric mean for consecutive positive integers is the mean of the two numbers.

46. The geometric mean for two perfect squares is a positive integer.

47. The geometric mean for two positive integers is another integer.

48. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate geometric mean.

a. **Tabular** Copy and complete the table of five ordered pairs (x, y) such that $\sqrt{xy} = 8$.

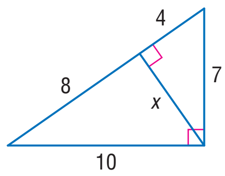
x	y	$\sqrt{xy}$
		8
		8
		8
		8
		8

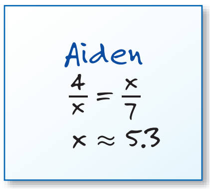
b. **Graphical** Graph the ordered pairs from your table in a scatter plot.

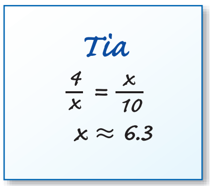
c. **Verbal** Make a conjecture as to the type of graph that would be formed if you connected the points from your scatter plot. Do you think the graph of any set of ordered pairs that results in the same geometric mean would have the same general shape? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

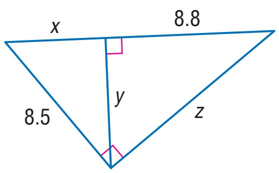
49. **ERROR ANALYSIS** Aiden and Tia are finding the value x in the triangle shown. Is either of them correct? Explain your reasoning.





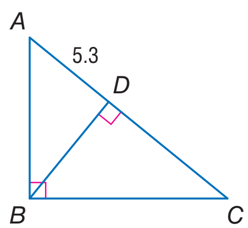


50. **CHALLENGE** Refer to the figure at the right. Find x, y, and z.



51. **OPEN ENDED** Find two pairs of whole numbers with a geometric mean that is also a whole number. What condition must be met in order for a pair of numbers to produce a whole-number geometric mean?

52. **CCSS REASONING** Refer to the figure at the right. The orthocenter of $\triangle ABC$ is located 6.4 units from point D. Find BC.



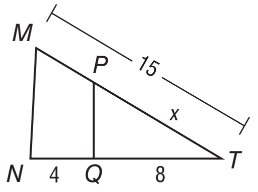
53. **WRITING IN MATH** Compare and contrast the arithmetic and geometric means of two numbers. When will the two means be equal? Justify your reasoning.

Standardized Test Practice

54. What is the geometric mean of 8 and 22 in simplest form?

A $4\sqrt{11}$ **C** $16\sqrt{11}$
B 15 **D** 176

55. SHORT RESPONSE If $\overline{MN} \parallel \overline{PQ}$, use a proportion to find the value of x. Show your work.

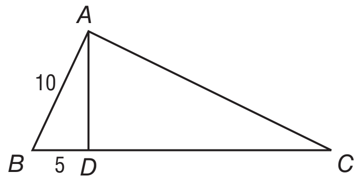


56. ALGEBRA What are the solutions of the quadratic equation $x^2 - 20 = 8x$?

F 2, 10 **H** −1, 20
G 20, 1 **J** −2, 10

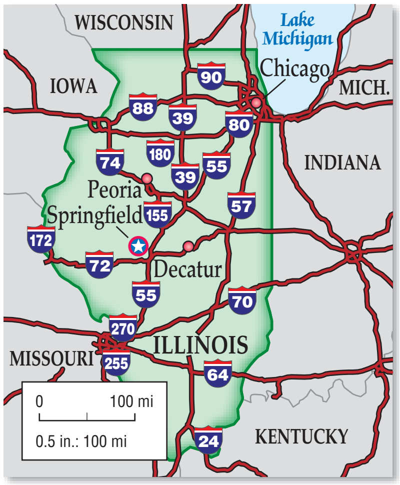
57. SAT/ACT In the figure, $\overline{AD}$ is perpendicular to $\overline{BC}$, and $\overline{AB}$ is perpendicular to $\overline{AC}$. What is BC?

A $5\sqrt{2}$
B $5\sqrt{3}$
C 20
D 25
E 75



Spiral Review

58. MAPS Use the map to estimate how long it would take to drive from Chicago to Springfield if you averaged 65 miles per hour. (Lesson 7-7)



Graph the original figure and its dilated image. Then verify that the dilation is a similarity transformation. (Lesson 7-6)

59. $A(-3, 1)$, $B(9, 7)$, $C(3, -2)$; $D(-1, 1)$, $E(3, 3)$, $F(1, 0)$

60. $G(-4, -4)$, $H(-1, 2)$, $J(2, -1)$; $K(-3, -2)$, $L(1, 0)$

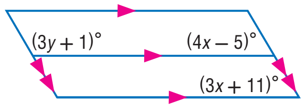
61. $M(7, -4)$, $N(5, -4)$, $P(7, -1)$; $Q(2, -8)$, $R(6, -8)$, $S(2, -2)$

The interior angle measure of a regular polygon is given. Identify the polygon. (Lesson 6-1)

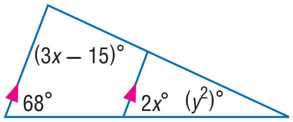
62. 108 **63.** 135

Find x and y in each figure. (Lesson 3-2)

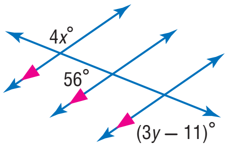
64.



65.

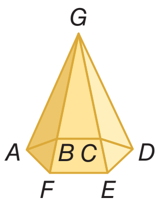


66.



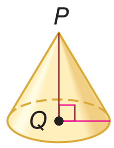
Identify each solid. Name the bases, faces, edges, and vertices. (Lesson 1-7)

67.



68.

69.



Skills Review

Simplify each expression by rationalizing the denominator.

70. $\frac{2}{\sqrt{2}}$ **71.** $\frac{16}{\sqrt{3}}$ **72.** $\frac{\sqrt{6}}{\sqrt{4}}$ **73.** $\frac{3\sqrt{5}}{\sqrt{11}}$ **74.** $\frac{21}{\sqrt{3}}$