LESSON 8-2 The Pythagorean Theorem and Its Converse

Then

- You used the Pythagorean Theorem to develop the Distance Formula.

Now

1. Use the Pythagorean Theorem.
2. Use the Converse of the Pythagorean Theorem.

Why?

- Tether lines are used to steady an inflatable snowman. Suppose you know the height at which the tether lines are attached to the snowman and how far away you want to anchor the tether in the ground. You can use the converse of the Pythagorean Theorem to adjust the lengths of the tethers to keep the snowman perpendicular to the ground.

NewVocabulary

Pythagorean triple

Common Core State Standards

Content Standards

G.SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★

G.MG.3 Apply geometric methods to solve problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

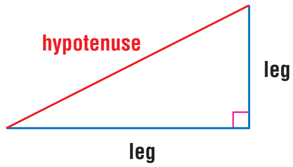
Mathematical Practices

1 Make sense of problems and persevere in solving them.

4 Model with mathematics.

1 The Pythagorean Theorem

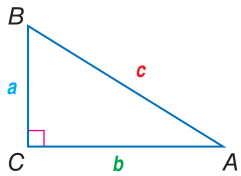
The Pythagorean Theorem is perhaps one of the most famous theorems in mathematics. It relates the lengths of the hypotenuse (side opposite the right angle) and legs (sides adjacent to the right angle) of a right triangle.



Theorem 8.4 Pythagorean Theorem

Words In a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

Symbols If $\triangle ABC$ is a right triangle with right angle C, then $a^2 + b^2 = c^2$.

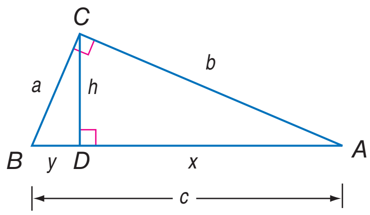


The geometric mean can be used to prove the Pythagorean Theorem.

Proof Pythagorean Theorem

Given: $\triangle ABC$ with right angle at C

Prove: $a^2 + b^2 = c^2$



Proof:

Draw right triangle ABC so C is the right angle. Then draw the altitude from C to $\overline{AB}$. Let $AB = c$, $AC = b$, $BC = a$, $AD = x$, $DB = y$, and $CD = h$. Two geometric means now exist.

$\frac{c}{a} = \frac{a}{y}$	and	$\frac{c}{b} = \frac{b}{x}$	Geometric Mean (Leg) Theorem
$a^2 = cy$		$b^2 = cx$	Cross products

$a^2 + b^2 = cy + cx$ — Add the equations.

$a^2 + b^2 = c(y + x)$ — Factor.

$a^2 + b^2 = c \cdot c$ — Since $c = y + x$, substitute c for $(y + x)$.

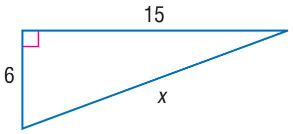
$a^2 + b^2 = c^2$ — Simplify.

You can use the Pythagorean Theorem to find the measure of any side of a right triangle given the lengths of the other two sides.

Example 1 Find Missing Measures Using the Pythagorean Theorem

Find x.

a.



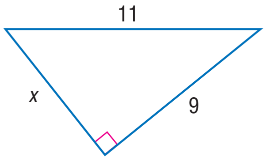
The side opposite the right angle is the hypotenuse, so $c = x$.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$6^2 + 15^2 = x^2$	$a = 6$ and $b = 15$
$261 = x^2$	Simplify.
$\sqrt{261} = x$	Take the positive square root of each side.
$3\sqrt{29} = x$	Simplify.

StudyTip

Positive Square Root When finding the length of a side using the Pythagorean Theorem, use only the positive and not the negative square root, since length cannot be negative.

b.

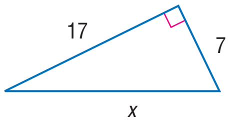


The hypotenuse is 11, so $c = 11$.

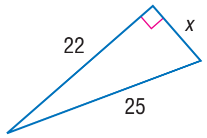
$a^2 + b^2 = c^2$	Pythagorean Theorem
$x^2 + 9^2 = 11^2$	$a = x$ and $b = 9$
$x^2 + 81 = 121$	Simplify.
$x^2 = 40$	Subtract 81 from each side.
$x = \sqrt{40}$ or $2\sqrt{10}$	Take the positive square root of each side and simplify.

GuidedPractice

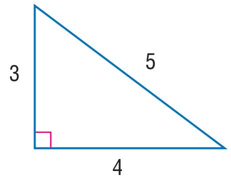
1A.



1B.



A **Pythagorean triple** is a set of three nonzero whole numbers a, b, and c, such that $a^2 + b^2 = c^2$. One common Pythagorean triple is 3, 4, 5; that is, the sides of a right triangle are in the ratio 3:4:5. The most common Pythagorean triples are shown below in the first row. The triples below these are found by multiplying each number in the triple by the same factor.



StudyTip

Pythagorean Triples If the measures of the sides of any right triangle are *not* whole numbers, the measures do not form a Pythagorean triple.

KeyConcept Common Pythagorean Triples

3, 4, 5	5, 12, 13	8, 15, 17	7, 24, 25
6, 8, 10	10, 24, 26	16, 30, 34	14, 48, 50
9, 12, 15	15, 36, 39	24, 45, 51	21, 72, 75
$3x$, $4x$, $5x$	$5x$, $12x$, $13x$	$8x$, $15x$, $17x$	$7x$, $24x$, $25x$

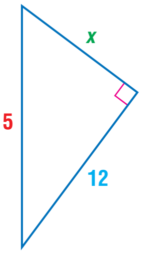
The largest number in each triple is the length of the hypotenuse.

Reading Math

3-4-5 A right triangle with side lengths 3, 4, and 5 is called a *3-4-5 right triangle.*

Example 2 Use a Pythagorean Triple

Use a Pythagorean triple to find x. Explain your reasoning.



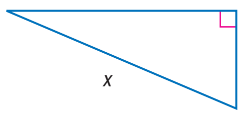
Notice that 15 and 12 are both multiples of 3, because $15 = 3 \cdot 5$ and $12 = 3 \cdot 4$. Since 3, 4, 5 is a Pythagorean triple, the missing leg length x is $3 \cdot 3$ or 9.

CHECK $12^2 + 9^2 \stackrel{?}{=} 15^2$ Pythagorean Theorem

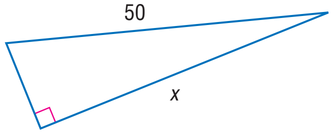
$225 = 225$ ✓ Simplify.

Guided Practice

2A.



2B.



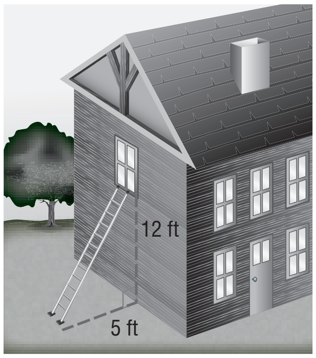
The Pythagorean Theorem can be used to solve many real-world problems.

Standardized Test Example 3 Use the Pythagorean Theorem

Damon is locked out of his house. The only open window is on the second floor, which is 12 feet above the ground. He needs to borrow a ladder from his neighbor. If he must place the ladder 5 feet from the house to avoid some bushes, what length of ladder does Damon need?

A 7 feet **C** 13 feet

B 11 feet **D** 17 feet



Note: Not drawn to scale

Read the Test Item

The distance the ladder is from the house, the height the ladder reaches, and the length of the ladder itself make up the lengths of the sides of a right triangle. You need to find the length of the ladder, which is the hypotenuse.

Solve the Test Item

Method 1 Use a Pythagorean triple.

The lengths of the legs are 5 and 12. 5, 12, 13 is a Pythagorean triple, so the length of the ladder is 13 feet.

Method 2 Use the Pythagorean Theorem.

Let x represent the length of the ladder.

$5^2 + 12^2 = x^2$ Pythagorean Theorem

$169 = x^2$ Simplify.

$\sqrt{169} = x$ Take the positive square root of each side.

$13 = x$ Simplify.

So, the answer is choice C.

Test-Taking Tip

CCSS Sense-Making

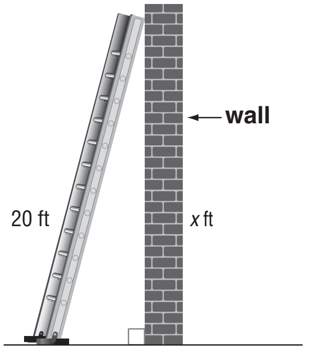
Since the hypotenuse of a right triangle is always the longest side, the length of the ladder in Example 3 must be greater than 5 or 12 feet. Since 7 and 11 feet are both less than 12 feet, choices A and B can be eliminated.

GuidedPractice

3. According to your company's safety regulations, the distance from the base of a ladder to a wall that it leans against should be at least one fourth of the ladder's total length. You are given a 20-foot ladder to place against a wall at a job site. If you follow the company's safety regulations, what is the maximum distance x up the wall the ladder will reach, to the nearest tenth?

F 12 feet **H** 20.6 feet

G 19.4 feet **J** 30.6 feet



Note: Not drawn to scale.

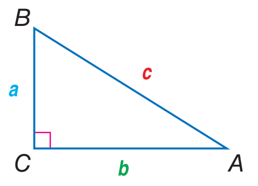
2 Converse of the Pythagorean Theorem

The converse of the Pythagorean Theorem also holds. You can use this theorem to help you determine whether a triangle is a right triangle given the measures of all three sides.

Theorem 8.5 Converse of the Pythagorean Theorem

Words If the sum of the squares of the lengths of the shortest sides of a triangle is equal to the square of the length of the longest side, then the triangle is a right triangle.

Symbols If $a^2 + b^2 = c^2$, then $\triangle ABC$ is a right triangle.



You will prove Theorem 8.5 in Exercise 35.

StudyTip

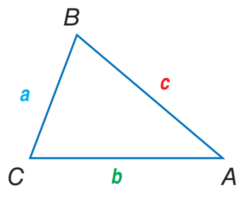
Determining the Longest Side If the measures of any of the sides of a triangle are expressed as radicals, you may wish to use a calculator to determine which length is the longest.

You can also use side lengths to classify a triangle as acute or obtuse.

Theorems Pythagorean Inequality Theorems

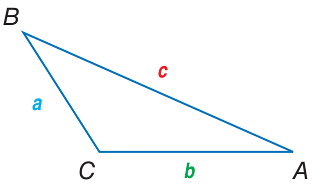
8.6 If the square of the length of the longest side of a triangle is less than the sum of the squares of the lengths of the other two sides, then the triangle is an acute triangle.

Symbols If $c^2 < a^2 + b^2$, then $\triangle ABC$ is acute.



8.7 If the square of the length of the longest side of a triangle is greater than the sum of the squares of the lengths of the other two sides, then the triangle is an obtuse triangle.

Symbols If $c^2 > a^2 + b^2$, then $\triangle ABC$ is obtuse.



You will prove Theorems 8.6 and 8.7 in Exercises 36 and 37, respectively.

Example 4 Classify Triangles

Determine whether each set of numbers can be the measures of the sides of a triangle. If so, classify the triangle as *acute*, *right*, or *obtuse*. Justify your answer.

a. 7, 14, 16

Step 1 Determine whether the measures can form a triangle using the Triangle Inequality Theorem.

$7 + 14 > 16$ ✔ $14 + 16 > 7$ ✔ $7 + 16 > 14$ ✔

The side lengths 7, 14, and 16 can form a triangle.

Step 2 Classify the triangle by comparing the square of the longest side to the sum of the squares of the other two sides.

$c^2 \stackrel{?}{=} a^2 + b^2$	Compare c^2 and $a^2 + b^2$.
$16^2 \stackrel{?}{=} 7^2 + 14^2$	Substitution
$256 > 245$	Simplify and compare.

Since $c^2 > a^2 + b^2$, the triangle is obtuse.

b. 9, 40, 41

Step 1 Determine whether the measures can form a triangle.

$9 + 40 > 41$ ✔ $40 + 41 > 9$ ✔ $9 + 41 > 40$ ✔

The side lengths 9, 40, and 41 can form a triangle.

Step 2 Classify the triangle.

$c^2 \stackrel{?}{=} a^2 + b^2$	Compare c^2 and $a^2 + b^2$.
$41^2 \stackrel{?}{=} 9^2 + 40^2$	Substitution
$1681 = 1681$	Simplify and compare.

Since $c^2 = a^2 + b^2$, the triangle is a right triangle.

GuidedPractice

4A. 11, 60, 61 **4B.** $2\sqrt{3}, 4\sqrt{2}, 3\sqrt{5}$ **4C.** 6.2, 13.8, 20

ReviewVocabulary

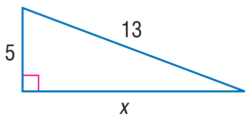
Triangle Inequality Theorem The sum of the lengths of any two sides of a triangle must be greater than the length of the third side.

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Example 1 Find x.

1.



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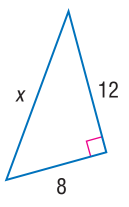

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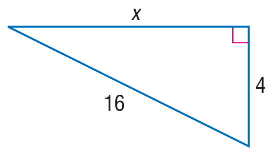
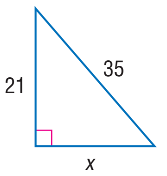


Example 2

4. Use a Pythagorean triple to find x. Explain your reasoning.

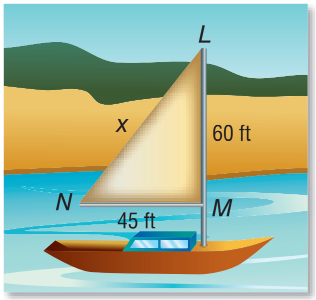


Example 3

5. **MULTIPLE CHOICE** The mainsail of a boat is shown. What is the length, in feet, of $\overline{LN}$?

A 52.5 C 72.5

B 65 D 75



Example 4

Determine whether each set of numbers can be the measures of the sides of a triangle. If so, classify the triangle as *acute*, *obtuse*, or *right*. Justify your answer.

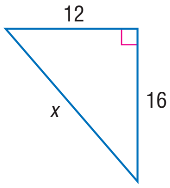
6. 15, 36, 39 7. 16, 18, 26 8. 15, 20, 24

Practice and Problem Solving

Extra Practice is on page R8.

Example 1

Find x.

9.


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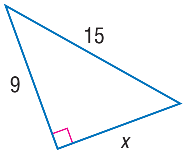

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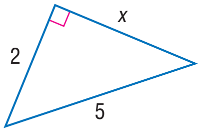

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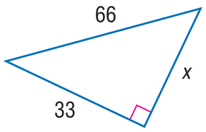

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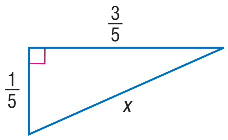

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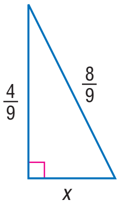
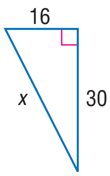


Example 2

CCSS PERSEVERANCE **Use a Pythagorean Triple to find x.**

15.


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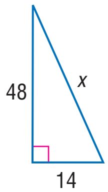

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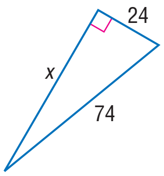

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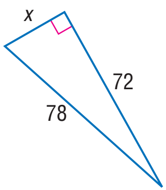
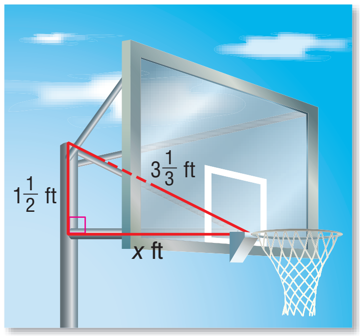
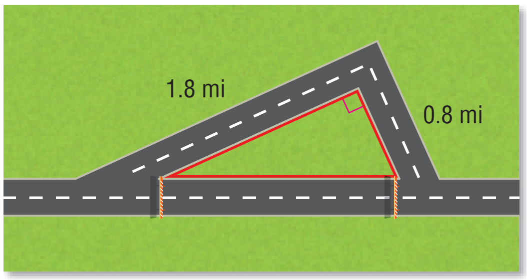


Example 3

19. **BASKETBALL** The support for a basketball goal forms a right triangle as shown. What is the length x of the horizontal portion of the support?



20. **DRIVING** The street that Khaliah usually uses to get to school is under construction. She has been taking the detour shown. If the construction starts at the point where Khaliah leaves her normal route and ends at the point where she re-enters her normal route, about how long is the stretch of road under construction?



Example 4 **Determine whether each set of numbers can be the measures of the sides of a triangle. If so, classify the triangle as *acute*, *obtuse*, or *right*. Justify your answer.**

21. 7, 15, 21

22. 10, 12, 23

23. 4.5, 20, 20.5

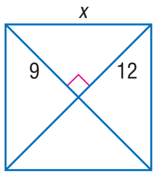
24. 44, 46, 91

25. 4.2, 6.4, 7.6

26. 4, 12, 14

Find x.

27.



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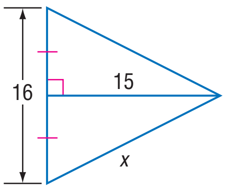

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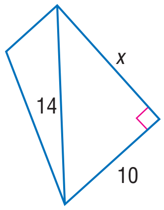


COORDINATE GEOMETRY **Determine whether $\triangle XYZ$ is an *acute*, *right*, or *obtuse* triangle for the given vertices. Explain.**

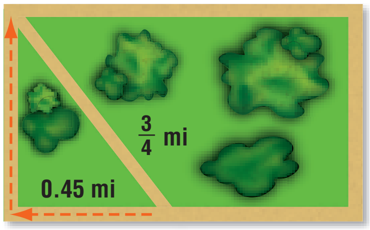
30. $X(-3, -2)$, $Y(-1, 0)$, $Z(0, -1)$

31. $X(-7, -3)$, $Y(-2, -5)$, $Z(-4, -1)$

32. $X(1, 2)$, $Y(4, 6)$, $Z(6, 6)$

33. $X(3, 1)$, $Y(3, 7)$, $Z(11, 1)$

34. **JOGGING** Brett jogs in the park three times a week. Usually, he takes a $\frac{3}{4}$-mile path that cuts through the park. Today, the path is closed, so he is taking the orange route shown. How much farther will he jog on his alternate route than he would have if he had followed his normal path?



35. **PROOF** Write a paragraph proof of Theorem 8.5.

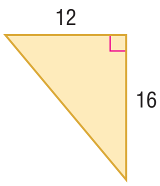
PROOF **Write a two-column proof for each theorem.**

36. Theorem 8.6

37. Theorem 8.7

CCSS SENSE-MAKING **Find the perimeter and area of each figure.**

38.



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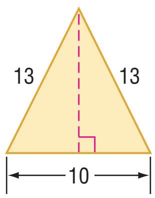

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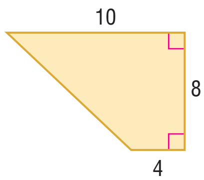
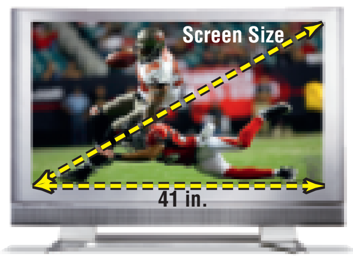


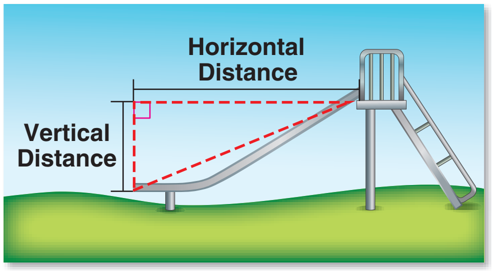
41. **ALGEBRA** The sides of a triangle have lengths x, $x + 5$, and 25. If the length of the longest side is 25, what value of x makes the triangle a right triangle?

42. **ALGEBRA** The sides of a triangle have lengths $2x$, 8, and 12. If the length of the longest side is $2x$, what values of x make the triangle acute?

43 **TELEVISION** The screen aspect ratio, or the ratio of the width to the height, of a high-definition television is 16:9. The size of a television is given by the diagonal distance across the screen. If an HDTV is 41 inches wide, what is its screen size?

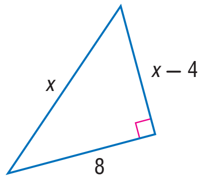


44. **PLAYGROUND** According to the *Handbook for Public Playground Safety*, the ratio of the vertical distance to the horizontal distance covered by a slide should not be more than about 4 to 7. If the horizontal distance allotted in a slide design is 14 feet, approximately how long should the slide be?



Find x.

45



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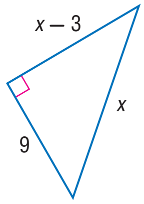

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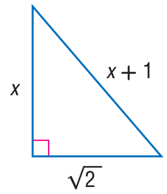


48. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate special right triangles.

a. Geometric Draw three different isosceles right triangles that have whole-number side lengths. Label the triangles ABC, MNP, and XYZ with the right angle located at vertex A, M, and X, respectively. Label the leg lengths of each side, and find the exact length of the hypotenuse.

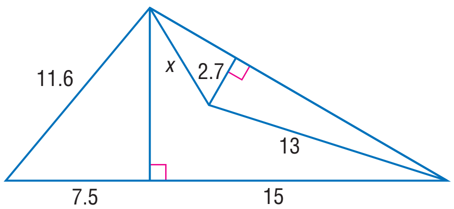
b. Tabular Copy and complete the table below.

Triangle	Length				Ratio	
ABC	BC		AB		$\frac{BC}{AB}$	
MNP	NP		MN		$\frac{NP}{MN}$	
XYZ	YZ		XY		$\frac{YZ}{XY}$	

c. Verbal Make a conjecture about the ratio of the hypotenuse to a leg of an isosceles right triangle.

H.O.T. Problems Use Higher-Order Thinking Skills

49. **CHALLENGE** Find the value of x in the figure at the right.



50. **CCSS ARGUMENTS** *True* or *false*? Any two right triangles with the same hypotenuse have the same area. Explain your reasoning.

51. **OPEN ENDED** Draw a right triangle with side lengths that form a Pythagorean triple. If you double the length of each side, is the resulting triangle *acute*, *right*, or *obtuse*? if you halve the length of each side? Explain.

52. **WRITING IN MATH** Research *incommensurable magnitudes*, and describe how this phrase relates to the use of irrational numbers in geometry. Include one example of an irrational number used in geometry.

Standardized Test Practice

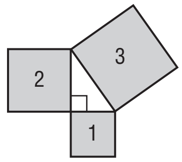
53. Which set of numbers cannot be the measures of the sides of a triangle?

A 10, 11, 20 **C** 35, 45, 75
B 14, 16, 28 **D** 41, 55, 98

54. A square park has a diagonal walkway from one corner to another. If the walkway is 120 meters long, what is the approximate length of each side of the park?

F 60 m **H** 170 m
G 85 m **J** 240 m

55. SHORT RESPONSE If the perimeter of square 2 is 200 units and the perimeter of square 1 is 150 units, what is the perimeter of square 3?



56. SAT/ACT In $\triangle ABC$, $\angle B$ is a right angle and $\angle A$ is 20° greater than $\angle C$. What is the measure of $\angle C$?

A 30 **C** 40 **E** 70
B 35 **D** 45

Spiral Review

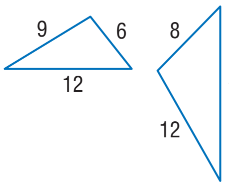
Find the geometric mean between each pair of numbers. (Lesson 8-1)

57. 9 and 4 **58.** 45 and 5 **59.** 12 and 15 **60.** 36 and 48

61. SCALE DRAWING Teodoro is creating a scale model of a skateboarding ramp on a 10-by-8-inch sheet of graph paper. If the real ramp is going to be 12 feet by 8 feet, find an appropriate scale for the drawing and determine the ramp's dimensions. (Lesson 7-7)

Determine whether the triangles are similar. If so, write a similarity statement. If not, what would be sufficient to prove the triangles similar? Explain your reasoning. (Lesson 7-3)

62.



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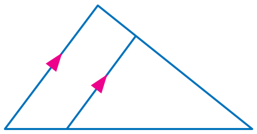

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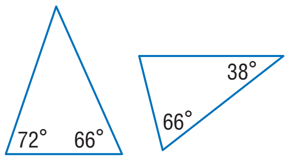
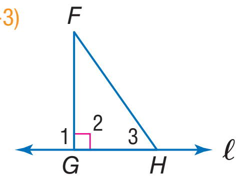


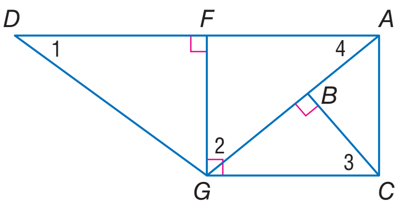
65. PROOF Write a two-column proof. (Lesson 5-3)

Given: $\overline{FG} \perp \ell$
$\overline{FH}$ is any nonperpendicular segment from F to ℓ.

Prove: $FH > FG$



Find each measure if $m\angle DGF = 53$ and $m\angle AGC = 40$. (Lesson 4-2)



66. $m\angle 1$ **67.** $m\angle 2$

68. $m\angle 3$ **69.** $m\angle 4$

Find the distance between each pair of parallel lines with the given equations. (Lesson 3-6)

70. $y = 4x$
$y = 4x - 17$

71. $y = 2x - 3$
$2x - y = -4$

72. $y = -0.75x - 1$
$3x + 4y = 20$

Skills Review

Find the value of x.

73. $18 = 3x\sqrt{3}$ **74.** $24 = 2x\sqrt{2}$ **75.** $9\sqrt{2} \cdot x = 18\sqrt{2}$ **76.** $2 = x \cdot \frac{4}{\sqrt{3}}$