LESSON

# 8-4 Trigonometry

**Then**

- You used the Pythagorean Theorem to find missing lengths in right triangles.

**Now**

1. Find trigonometric ratios using right triangles.
2. Use trigonometric ratios to find angle measures in right triangles.

**Why?**

- The steepness of a hiking trail is often expressed as a *percent of grade*. The steepest part of Bright Angel Trail in the Grand Canyon National Park has about a 15.7% grade. This means that the trail rises or falls 15.7 feet over a horizontal distance of 100 feet. You can use trigonometric ratios to determine that this steepness is equivalent to an angle of about 9°.

**NewVocabulary**

trigonometry
trigonometric ratio
sine
cosine
tangent
inverse sine
inverse cosine
inverse tangent

**Common Core State Standards**

**Content Standards**

**G.SRT.6** Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

**G.SRT.7** Explain and use the relationship between the sine and cosine of complementary angles.

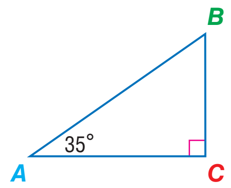
**Mathematical Practices**

1 Make sense of problems and persevere in solving them.

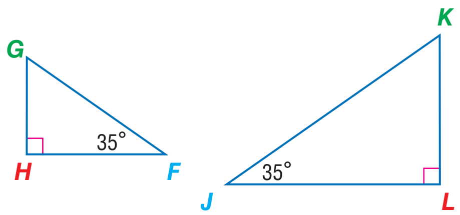
5 Use appropriate tools strategically.

## 1 Trigonometric Ratios

The word **trigonometry** comes from two Greek terms, *trigon*, meaning triangle, and *metron*, meaning measure. The study of trigonometry involves triangle measurement. A **trigonometric ratio** is a ratio of the lengths of two sides of a right triangle. One trigonometric ratio of $\triangle ABC$ is $\frac{AC}{AB}$.




By AA Similarity, a right triangle with a given acute angle measure is similar to every other right triangle with the same acute angle measure. So, trigonometric ratios are constant for a given angle measure.

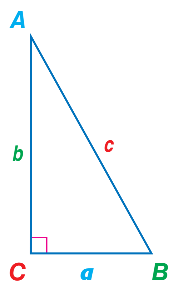



$\triangle ABC \sim \triangle FGH \sim \triangle JKL$, so $\frac{AC}{AB} = \frac{FH}{FG} = \frac{JL}{JK}$

The names of the three most common trigonometric ratios are given below.

**KeyConcept Trigonometric Ratios**

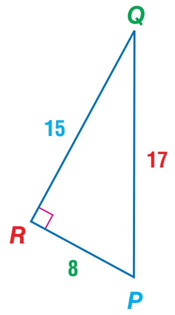
| Words | Symbols |
|---|---|
| If $\triangle ABC$ is a right triangle with acute $\angle A$, then the **sine** of $\angle A$ (written sin $A$) is the ratio of the length of the leg opposite $\angle A$ (opp) to the length of the hypotenuse (hyp). | $\sin A = \frac{\text{opp}}{\text{hyp}}$ or $\frac{a}{c}$<br>$\sin B = \frac{\text{opp}}{\text{hyp}}$ or $\frac{b}{c}$ |
| If $\triangle ABC$ is a right triangle with acute $\angle A$, then the **cosine** of $\angle A$ (written cos $A$) is the ratio of the length of the leg adjacent $\angle A$ (adj) to the length of the hypotenuse (hyp). | $\cos A = \frac{\text{adj}}{\text{hyp}}$ or $\frac{b}{c}$<br>$\cos B = \frac{\text{adj}}{\text{hyp}}$ or $\frac{a}{c}$ |
| If $\triangle ABC$ is a right triangle with acute $\angle A$, then the **tangent** of $\angle A$ (written tan $A$) is the ratio of the length of the leg opposite $\angle A$ (opp) to the length of the leg adjacent $\angle A$ (adj). | $\tan A = \frac{\text{opp}}{\text{adj}}$ or $\frac{a}{b}$<br>$\tan B = \frac{\text{opp}}{\text{adj}}$ or $\frac{b}{a}$ |




Douglas Peebles Photography/Alamy

## Example 1 Find Sine, Cosine, and Tangent Ratios

**Express each ratio as a fraction and as a decimal to the nearest hundredth.**




**a.** sin $P$

$\sin P = \dfrac{\text{opp}}{\text{hyp}}$

$= \dfrac{15}{17}$ or about 0.88

**b.** cos $P$

$\cos P = \dfrac{\text{adj}}{\text{hyp}}$

$= \dfrac{8}{17}$ or about 0.47

**c.** tan $P$

$\tan P = \dfrac{\text{opp}}{\text{adj}}$

$= \dfrac{15}{8}$ or about 1.88

**d.** sin $Q$

$\sin Q = \dfrac{\text{opp}}{\text{hyp}}$

$= \dfrac{8}{17}$ or about 0.47

**e.** cos $Q$

$\cos Q = \dfrac{\text{adj}}{\text{hyp}}$

$= \dfrac{15}{17}$ or about 0.88

**f.** tan $Q$

$\tan Q = \dfrac{\text{opp}}{\text{adj}}$

$= \dfrac{8}{15}$ or about 0.53

### StudyTip

**Memorizing Trigonometric Ratios** SOH-CAH-TOA is a mnemonic device for learning the ratios for sine, cosine, and tangent using the first letter of each word in the ratios.

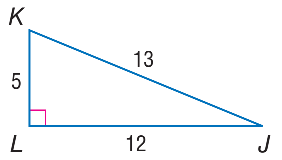
$\sin A = \dfrac{\text{opp}}{\text{hyp}}$

$\cos A = \dfrac{\text{adj}}{\text{hyp}}$

$\tan A = \dfrac{\text{opp}}{\text{adj}}$

### GuidedPractice

**1.** Find sin $J$, cos $J$, tan $J$, sin $K$, cos $K$, and tan $K$. Express each ratio as a fraction and as a decimal to the nearest hundredth.




Special right triangles can be used to find the sine, cosine, and tangent of 30°, 60°, and 45° angles.

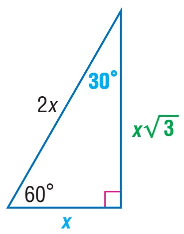
## Example 2 Use Special Right Triangles to Find Trigonometric Ratios

**Use a special right triangle to express the tangent of 30° as a fraction and as a decimal to the nearest hundredth.**

Draw and label the side lengths of a 30°-60°-90° right triangle, with $x$ as the length of the shorter leg.

The side opposite the 30° angle has a measure of $x$.

The side adjacent to the 30° angle has a measure of $x\sqrt{3}$.




$\tan 30° = \dfrac{\text{opp}}{\text{adj}}$ — Definition of tangent ratio

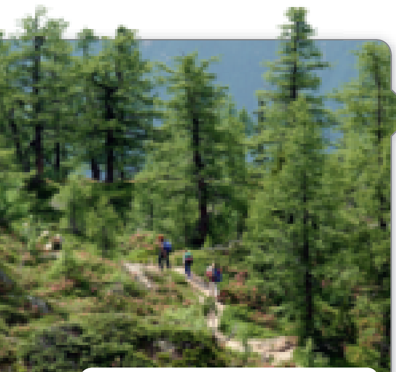
$= \dfrac{\cancel{x}}{\cancel{x}\sqrt{3}}$ — Substitution

$= \dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$ — Simplify and rationalize the denominator.

$= \dfrac{\sqrt{3}}{3}$ or about 0.58 — Simplify and use a calculator.

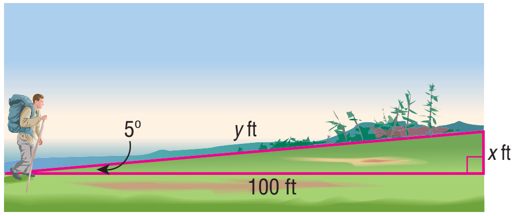
### GuidedPractice

**2.** Use a special right triangle to express the cosine of 45° as a fraction and as a decimal to the nearest hundredth.



## Real-World Example 3 Estimate Measures Using Trigonometry

**HIKING A certain part of a hiking trail slopes upward at about a 5° angle. After traveling a horizontal distance of 100 feet along this part of the trail, what would be the change in a hiker's vertical position? What distance has the hiker traveled along the path?**




Let $m\angle A = 5$. The vertical change in the hiker's position is $x$, the measure of the leg opposite $\angle A$. The horizontal distance traveled is 100 feet, the measure of the leg adjacent to $\angle A$. Since the length of the leg opposite and the leg adjacent to a given angle are involved, write an equation using a tangent ratio.

$\tan A = \frac{\text{opp}}{\text{adj}}$ Definition of tangent ratio

$\tan 5° = \frac{x}{100}$ Substitution

$100 \cdot \tan 5° = x$ Multiply each side by 100.

Use a calculator to find $x$.

100 [TAN] 5 [ENTER] **8.748866353**

The hiker is about 8.75 feet higher than when he started walking.

The distance $y$ traveled along the path is the length of the hypotenuse, so you can use a cosine ratio to find this distance.

$\cos A = \frac{\text{adj}}{\text{hyp}}$ Definition of cosine ratio

$\cos 5° = \frac{100}{y}$ Substitution

$y \cdot \cos 5° = 100$ Multiply each side by $y$.

$y = \frac{100}{\cos 5°}$ Divide each side by cos 5°.

Use a calculator to find $y$.

100 [÷] [COS] 5 [ENTER] **100.3819838**

The hiker has traveled a distance of about 100.38 feet along the path.

### Real-WorldLink

The grade of a trail often changes many times. Average grade is the average of several consecutive running grades of a trail. Maximum grade is the smaller section of a trail that exceeds the trail's typical running grade. Trails often have maximum grades that are much steeper than a trail's average running grade.

**Source:** Federal Highway Administration

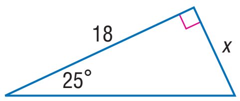
### StudyTip

**Graphing Calculator** Be sure your graphing calculator is in degree mode rather than radian mode.

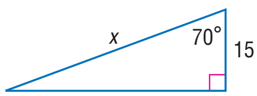
### GuidedPractice

**Find $x$ to the nearest hundredth.**

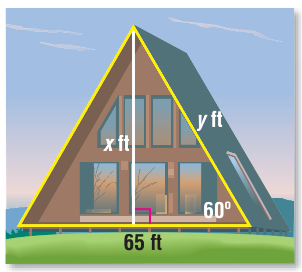
**3A.**




**3B.**




**3C. ARCHITECTURE** The front of the vacation cottage shown is an isosceles triangle. What is the height $x$ of the cottage above its foundation? What is the length $y$ of the roof? Explain your reasoning.




J. A. Kraulis/Masterfile

## 2 Use Inverse Trigonometric Ratios

In Example 2, you found that tan 30° ≈ 0.58. It follows that if the tangent of an acute angle is 0.58, then the angle measures approximately 30.

If you know the sine, cosine, or tangent of an acute angle, you can use a calculator to find the measure of the angle, which is the inverse of the trigonometric ratio.

**ReadingMath**

**Inverse Trigonometric Ratios** The expression $\sin^{-1} x$ is read *the inverse sine of x* and is interpreted as the angle with sine *x*. Be careful not to confuse this notation with the notation for negative exponents—
$\sin^{-1} x \neq \frac{1}{\sin x}$.
Instead, this notation is similar to the notation for an inverse function, $f^{-1}(x)$.

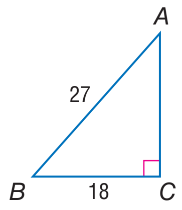
### KeyConcept Inverse Trigonometric Ratios

| | |
|---|---|
| **Words** | If $\angle A$ is an acute angle and the sine of *A* is *x*, then the **inverse sine** of *x* is the measure of $\angle A$. |
| **Symbols** | If $\sin A = x$, then $\sin^{-1} x = m\angle A$. |
| **Words** | If $\angle A$ is an acute angle and the cosine of *A* is *x*, then the **inverse cosine** of *x* is the measure of $\angle A$. |
| **Symbols** | If $\cos A = x$, then $\cos^{-1} x = m\angle A$. |
| **Words** | If $\angle A$ is an acute angle and the tangent of *A* is *x*, then the **inverse tangent** of *x* is the measure of $\angle A$. |
| **Symbols** | If $\tan A = x$, then $\tan^{-1} x = m\angle A$. |

So if tan 30° ≈ 0.58, then $\tan^{-1} 0.58 \approx 30°$.

### Example 4 Find Angle Measures Using Inverse Trigonometric Ratios

**Use a calculator to find the measure of ∠*A* to the nearest tenth.**




The measures given are those of the leg opposite ∠*A* and the hypotenuse, so write an equation using the sine ratio.

$\sin A = \frac{18}{27}$ or $\frac{2}{3}$ $\quad \sin A = \frac{\text{opp}}{\text{hyp}}$

If $\sin A = \frac{2}{3}$, then $\sin^{-1} \frac{2}{3} = m\angle A$. Use a calculator.

**KEYSTROKES:** [2nd] [SIN $^{-1}$] [ ( ] 2 [÷] 3 [ ) ] [ENTER] **41.8103149**

So, $m\angle A \approx 41.8°$.

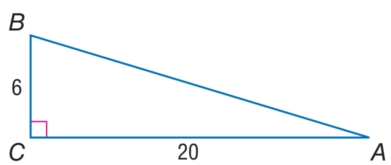
**StudyTip**

**CCSS Tools** Use a graphing calculator. The second functions of the [SIN], [COS], and [TAN] keys are usually the inverses.

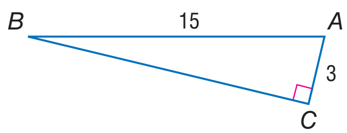
**GuidedPractice**

**Use a calculator to find the measure of ∠*A* to the nearest tenth.**

**4A.**




**4B.**

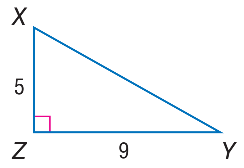


When you use given measures to find the unknown angle and side measures of a right triangle, this is known as *solving a right triangle*. To solve a right triangle, you need to know

- two side lengths or
- one side length and the measure of one acute angle.

## Example 5 Solve a Right Triangle

**Solve the right triangle. Round side measures to the nearest tenth and angle measures to the nearest degree.**




**Step 1** Find $m\angle X$ by using a tangent ratio.

| | |
|---|---|
| $\tan X = \frac{9}{5}$ | $\tan X = \frac{\text{opp}}{\text{adj}}$ |
| $\tan^{-1}\frac{9}{5} = m\angle X$ | Definition of inverse tangent |
| $60.9453959 \approx m\angle X$ | Use a calculator. |

So, $m\angle X \approx 61$.

**Step 2** Find $m\angle Y$ using Corollary 4.1, which states that the acute angles of a right triangle are complementary.

| | |
|---|---|
| $m\angle X + m\angle Y = 90$ | Corollary 4.1 |
| $61 + m\angle Y \approx 90$ | $m\angle X \approx 61$ |
| $m\angle Y \approx 29$ | Subtract 61 from each side. |

So, $m\angle Y \approx 29$.

**Step 3** Find $XY$ by using the Pythagorean Theorem.

| | |
|---|---|
| $(XZ)^2 + (ZY)^2 = (XY)^2$ | Pythagorean Theorem |
| $5^2 + 9^2 = (XY)^2$ | Substitution |
| $106 = (XY)^2$ | Simplify. |
| $\sqrt{106} = XY$ | Take the positive square root of each side. |
| $10.3 \approx XY$ | Use a calculator. |

So $XY \approx 10.3$.

### StudyTip

**Alternative Methods** Right triangles can often be solved using different methods. In Example 5, $m\angle Y$ could have been found using a tangent ratio, and $m\angle X$ and a sine ratio could have been used to find $XY$.

### WatchOut!

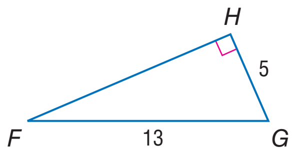
**Approximation** If using calculated measures to find other measures in a right triangle, be careful not to round values until the last step. So in the following equation, use $\tan^{-1}\frac{9}{5}$ instead of its approximate value, 61°.

$$XY = \frac{9}{\sin X}$$
$$= \frac{9}{\sin\left(\tan^{-1}\frac{9}{5}\right)}$$
$$\approx 10.3$$

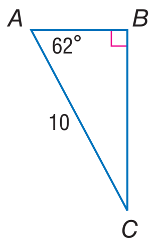
## Guided Practice

**Solve each right triangle. Round side measures to the nearest tenth and angle measures to the nearest degree.**

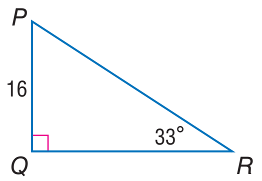
**5A.**




**5B.**




**5C.**



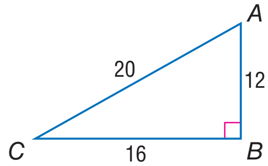
## Check Your Understanding

= Step-by-Step Solutions begin on page R14.

**Example 1** **Express each ratio as a fraction and as a decimal to the nearest hundredth.**

**1.** sin $A$ **2.** tan $C$ **3.** cos $A$

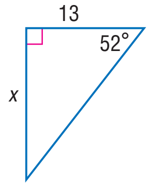
**4.** tan $A$ **5.** cos $C$ **6.** sin $C$



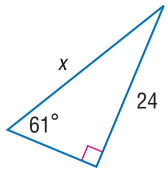

**Example 2** **7.** Use a special right triangle to express sin 60° as a fraction and as a decimal to the nearest hundredth.

**Example 3** **Find $x$. Round to the nearest hundredth.**

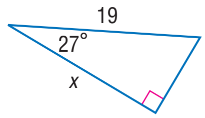
**8.**



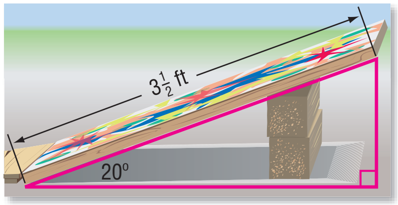

**9.**




**10.**

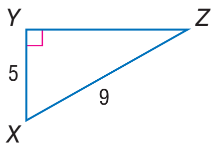



**11. SPORTS** David is building a bike ramp. He wants the angle that the ramp makes with the ground to be 20°. If the board he wants to use for his ramp is $3\frac{1}{2}$ feet long, about how tall will the ramp need to be at the highest point?

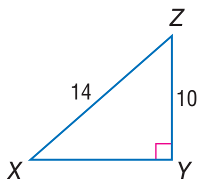



**Example 4** **CCSS TOOLS Use a calculator to find the measure of ∠$Z$ to the nearest tenth.**

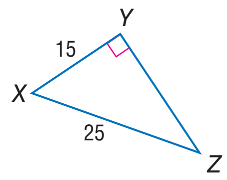
**12.**



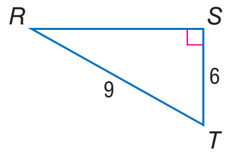

**13.**




**14.**




**Example 5** **15.** Solve the right triangle. Round side measures to the nearest tenth and angle measures to the nearest degree.

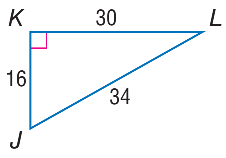



## Practice and Problem Solving

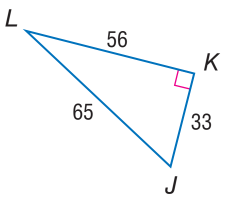
Extra Practice is on page R8.

**Example 1** **Find sin $J$, cos $J$, tan $J$, sin $L$, cos $L$, and tan $L$. Express each ratio as a fraction and as a decimal to the nearest hundredth.**

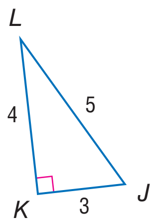
**16.**



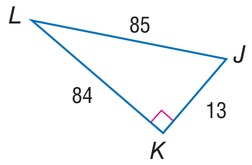

**17.**



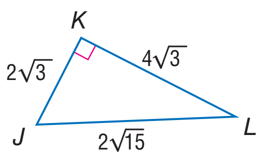

**18.**



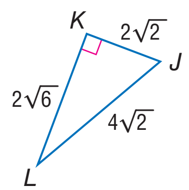

**19.**




**20.**




**21.**



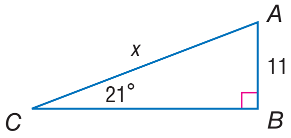
**Example 2** **Use a special right triangle to express each trigonometric ratio as a fraction and as a decimal to the nearest hundredth.**

**22.** tan 60° **23.** cos 30° **24.** sin 45°

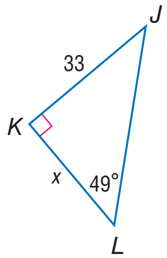
**25.** sin 30° **26.** tan 45° **27.** cos 60°

**Example 3** **Find $x$. Round to the nearest tenth.**

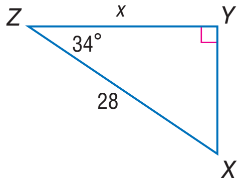
**28.**



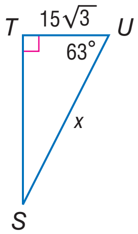

**29.**



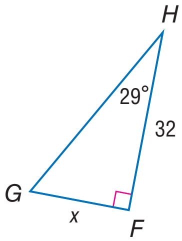

**30.**



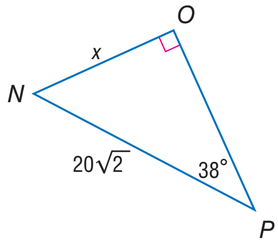

**31.**



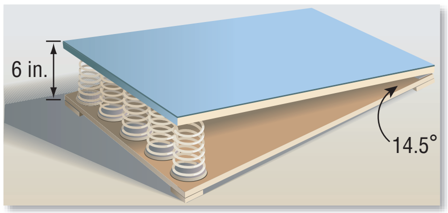

**32.**



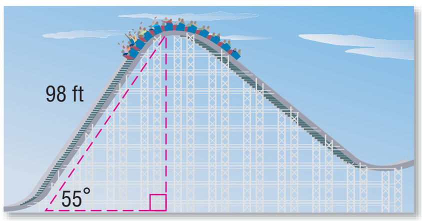

**33.**




**34. GYMNASTICS** The springboard that Eric uses in his gymnastics class has 6-inch coils and forms an angle of 14.5° with the base. About how long is the springboard?

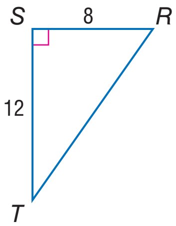



**35. ROLLER COASTERS** The angle of ascent of the first hill of a roller coaster is 55°. If the length of the track from the beginning of the ascent to the highest point is 98 feet, what is the height of the roller coaster when it reaches the top of the first hill?

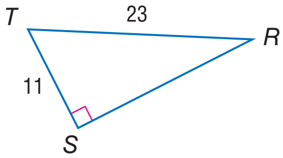



**Example 4** **CCSS TOOLS** **Use a calculator to find the measure of $\angle T$ to the nearest tenth.**

**36.**




**37.**

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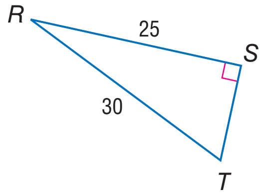


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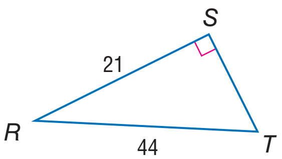
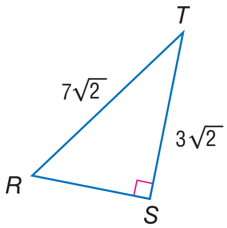
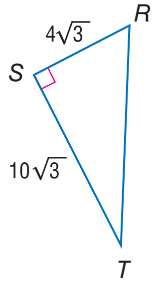



**40.**

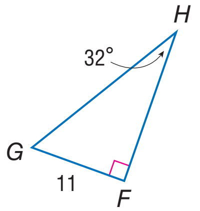



**41.**

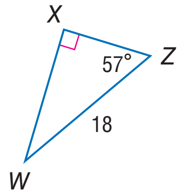


**Example 5** **Solve each right triangle. Round side measures to the nearest tenth and angle measures to the nearest degree.**

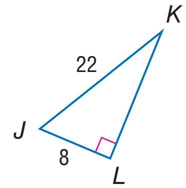
**42.**



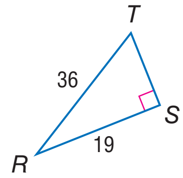

**43.**



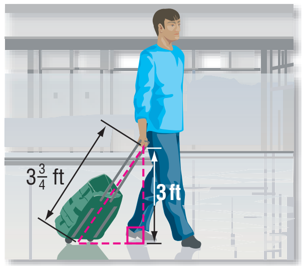

**44.**




**45.**




**46.** **BACKPACKS** Ramón has a rolling backpack that is $3\frac{3}{4}$ feet tall when the handle is extended. When he is pulling the backpack, Ramon's hand is 3 feet from the ground. What angle does his backpack make with the floor? Round to the nearest degree.




**COORDINATE GEOMETRY** **Find the measure of each angle to the nearest tenth of a degree using the Distance Formula and an inverse trigonometric ratio.**

**47.** $\angle K$ in right triangle $JKL$ with vertices $J(-2, -3)$, $K(-7, -3)$, and $L(-2, 4)$

**48.** $\angle Y$ in right triangle $XYZ$ with vertices $X(4, 1)$, $Y(-6, 3)$, and $Z(-2, 7)$

**49.** $\angle A$ in right triangle $ABC$ with vertices $A(3, 1)$, $B(3, -3)$, and $C(8, -3)$

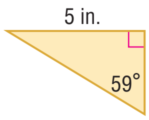
**50.** **SCHOOL SPIRIT** Hana is making a pennant for each of the 18 girls on her basketball team. She will use $\frac{1}{2}$-inch seam binding to finish the edges of the pennants.

**a.** What is the total length of seam binding needed to finish all of the pennants?

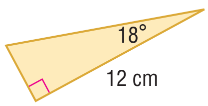
**b.** If seam binding is sold in 3-yard packages at a cost of $1.79, how much will it cost?


**CCSS SENSE-MAKING** **Find the perimeter and area of each triangle. Round to the nearest hundredth.**

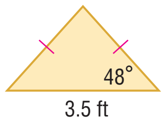
**51.**




**52.**



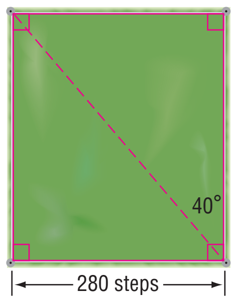

**53.**




**54.** Find the tangent of the greater acute angle in a triangle with side lengths of 3, 4, and 5 centimeters.

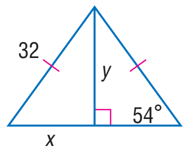
**55.** Find the cosine of the smaller acute angle in a triangle with side lengths of 10, 24, and 26 inches.

**56.** **ESTIMATION** Ethan and Tariq want to estimate the area of the field that their team will use for soccer practice. They know that the field is rectangular, and they have paced off the width of the field as shown. They used the fence posts at the corners of the field to estimate that the angle between the length of the field and the diagonal is about 40°. If they assume that each of their steps is about 18 inches, what is the area of the practice field in square feet? Round to the nearest square foot.

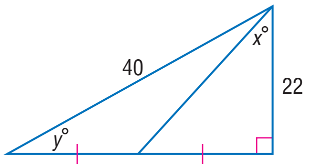


**Find $x$ and $y$. Round to the nearest tenth.**

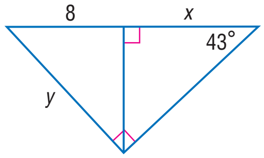
**57.**



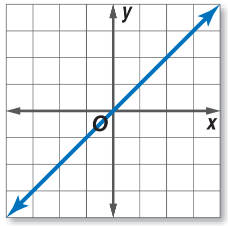

**58.**




**59.**




**60. COORDINATE GEOMETRY** Show that the slope of a line at 225° from the $x$-axis is equal to the tangent of 225°.




**61. MULTIPLE REPRESENTATIONS** In this problem, you will investigate an algebraic relationship between the sine and cosine ratios.

**a. Geometric** Draw three right triangles that are not similar to each other. Label the triangles $ABC$, $MNP$, and $XYZ$, with the right angles located at vertices $B$, $N$, and $Y$, respectively. Measure and label each side of the three triangles.

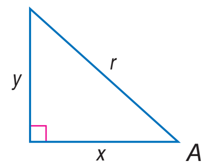
**b. Tabular** Copy and complete the table below.

| Triangle | Trigonometric Ratios | | | | Sum of Ratios Squared | |
|---|---|---|---|---|---|---|
| $ABC$ | $\cos A$ | | $\sin A$ | | $(\cos A)^2 + (\sin A)^2 =$ | |
| | $\cos C$ | | $\sin C$ | | $(\cos C)^2 + (\sin C)^2 =$ | |
| $MNP$ | $\cos M$ | | $\sin M$ | | $(\cos M)^2 + (\sin M)^2 =$ | |
| | $\cos P$ | | $\sin P$ | | $(\cos P)^2 + (\sin P)^2 =$ | |
| $XYZ$ | $\cos X$ | | $\sin X$ | | $(\cos X)^2 + (\sin X)^2 =$ | |
| | $\cos Z$ | | $\sin Z$ | | $(\cos Z)^2 + (\sin Z)^2 =$ | |

**c. Verbal** Make a conjecture about the sum of the squares of the cosine and sine of an acute angle of a right triangle.

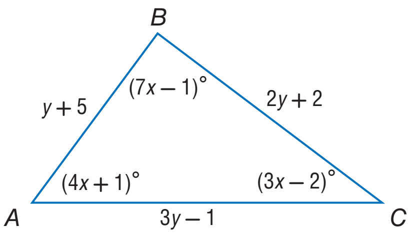
**d. Algebraic** Express your conjecture algebraically for an angle $X$.

**e. Analytical** Show that your conjecture is valid for angle $A$ in the figure at the right using the trigonometric functions and the Pythagorean Theorem.




## H.O.T. Problems Use Higher-Order Thinking Skills

**62. CHALLENGE** Solve $\triangle ABC$. Round to the nearest whole number.



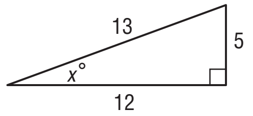

**63. REASONING** Are the values of sine and cosine for an acute angle of a right triangle always less than 1? Explain.

**64. CCSS REASONING** What is the relationship between the sine and cosine of complementary angles? Explain your reasoning and use the relationship to find cos 50 if sin 40 $\approx$ 0.64.

**65. WRITING IN MATH** Explain how you can use ratios of the side lengths to find the angle measures of the acute angles in a right triangle.

## Standardized Test Practice

**66.** What is the value of tan $x$?




**A** $\tan x = \frac{13}{5}$

**B** $\tan x = \frac{12}{5}$

**C** $\tan x = \frac{5}{13}$

**D** $\tan x = \frac{5}{12}$

**67. ALGEBRA** Which of the following has the same value as $2^{-12} \times 2^3$?

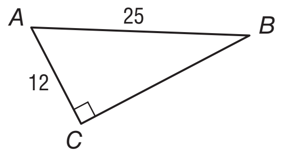
**F** $2^{-36}$

**G** $4^{-9}$

**H** $2^{-9}$

**J** $2^{-4}$

**68. GRIDDED RESPONSE** If $AC = 12$ and $AB = 25$, what is the measure of $\angle B$ to the nearest tenth?




**69. SAT/ACT** The area of a right triangle is 240 square inches. If the base is 30 inches long, how many inches long is the hypotenuse?

**A** 5

**B** 8

**C** 16

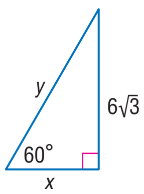
**D** $2\sqrt{241}$

**E** 34

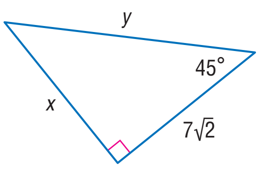
## Spiral Review

**Find $x$ and $y$.** (Lesson 8-3)

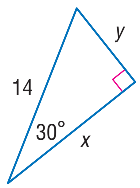
**70.**




**71.**




**72.**




**Determine whether each set of numbers can be the measures of the sides of a triangle. If so, classify the triangle as *acute*, *obtuse*, or *right*. Justify your answer.** (Lesson 8-2)

**73.** 8, 15, 17

**74.** 11, 12, 24

**75.** 13, 30, 35

**76.** 18, 24, 30

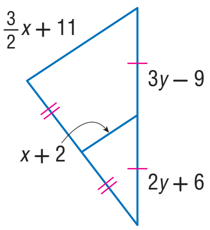
**77.** 3.2, 5.3, 8.6

**78.** $6\sqrt{3}$, 14, 17

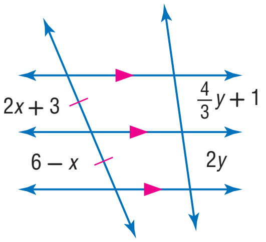
**79. MAPS** The scale on the map of New Mexico is 2 centimeters = 160 miles. The width of New Mexico through Albuquerque on the map is 4.1 centimeters. How long would it take to drive across New Mexico if you drove at an average of 60 miles per hour? (Lesson 7-7)


**ALGEBRA Find $x$ and $y$.** (Lesson 7-4)

**80.**




**81.**




## Skills Review

**Solve each proportion. Round to the nearest tenth if necessary.**

**82.** $2.14 = \frac{x}{12}$

**83.** $0.05x = 13$

**84.** $0.37 = \frac{32}{x}$

**85.** $0.74 = \frac{14}{x}$

**86.** $1.66 = \frac{x}{23}$

**87.** $0.21 = \frac{33}{x}$