LESSON 8-6 The Law of Sines and Law of Cosines

Then

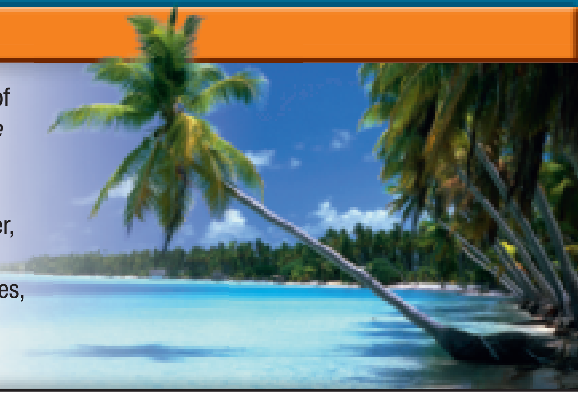
- You used trigonometric ratios to solve right triangles.

Now

1. Use the Law of Sines to solve triangles.
2. Use the Law of Cosines to solve triangles.

Why?

- You have learned that the height or length of a tree can be calculated using *right triangle trigonometry* if you know the angle of elevation to the top of the tree and your distance from the tree. Some trees, however, grow at an angle or lean due to weather damage. To calculate the length of such trees, you must use other forms of trigonometry.



NewVocabulary
Law of Sines
Law of Cosines

Common Core State Standards

Content Standards

G.SRT.9 Derive the formula $A = \frac{1}{2}ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G.SRT.10 Prove the Laws of Sines and Cosines and use them to solve problems.

Mathematical Practices

4 Model with mathematics.
1 Make sense of problems and persevere in solving them.

1 Law of Sines

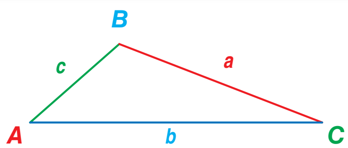
In Lesson 8-4, you used trigonometric ratios to find side lengths and acute angle measures in *right* triangles. To find measures for nonright triangles, the definitions of sine and cosine can be extended to obtuse angles.

The **Law of Sines** can be used to find side lengths and angle measures for any triangle.

Theorem 8.10 Law of Sines

If $\triangle ABC$ has lengths a, b, and c, representing the lengths of the sides opposite the angles with measures A, B, and C, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



You will prove one of the proportions for Theorem 8.10 in Exercise 45.

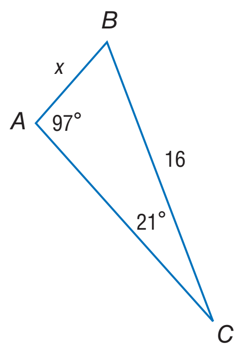
You can use the Law of Sines to solve a triangle if you know the measures of two angles and any side (AAS or ASA).

Example 1 Law of Sines (AAS)

Find x. Round to the nearest tenth.

We are given the measures of two angles and a nonincluded side, so use the Law of Sines to write a proportion.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines
$\frac{\sin 97°}{16} = \frac{\sin 21°}{x}$	$m\angle A = 97$, $a = 16$, $m\angle C = 21$, $c = x$
$x \sin 97° = 16 \sin 21°$	Cross Products Property
$x = \frac{16 \sin 21°}{\sin 97°}$	Divide each side by sin 97°.
$x \approx 5.8$	Use a calculator.



GuidedPractice

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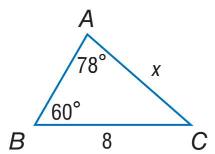
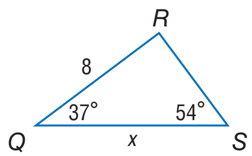


1B.

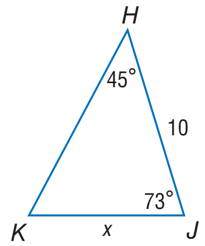


If given ASA, use the Triangle Angle Sum Theorem to first find the measure of the third angle.

Example 2 Law of Sines (ASA)

Find x. Round to the nearest tenth.

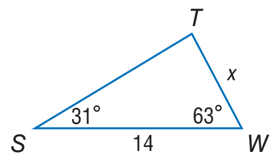
By the Triangle Angle Sum Theorem, $m\angle K = 180 - (45 + 73)$ or 62.



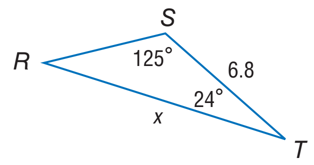
$\frac{\sin H}{h} = \frac{\sin K}{k}$	Law of Sines
$\frac{\sin 45°}{x} = \frac{\sin 62°}{10}$	$m\angle H = 45$, $h = x$, $m\angle K = 62$, $k = 10$
$10 \sin 45° = x \sin 62°$	Cross Products Property
$\frac{10 \sin 45°}{\sin 62°} = x$	Divide each side by sin 62°.
$x \approx 8.0$	Use a calculator.

Guided Practice

2A.



2B.



Study Tip

Ambiguous Case You can sometimes use the Law of Sines to solve a triangle if you know the measures of two sides and a nonincluded angle (SSA). However, these three measures do not always determine exactly one triangle. You will learn more about this *ambiguous case* in Extend 8-6.

2 Law of Cosines

When the Law of Sines cannot be used to solve a triangle, the Law of Cosines may apply.

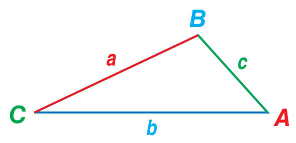
Theorem 8.11 Law of Cosines

If $\triangle ABC$ has lengths a, b, and c, representing the lengths of the sides opposite the angles with measures A, B, and C, then

$a^2 = b^2 + c^2 - 2bc \cos A$,

$b^2 = a^2 + c^2 - 2ac \cos B$, and

$c^2 = a^2 + b^2 - 2ab \cos C$.



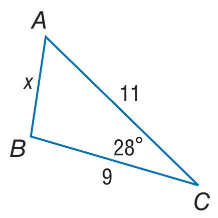
You will prove one of the equations for Theorem 8.11 in Exercise 46.

You can use the **Law of Cosines** to solve a triangle if you know the measures of two sides and the included angle (SAS).

Example 3 Law of Cosines (SAS)

Find x. Round to the nearest tenth.

We are given the measures of two sides and their included angle, so use the Law of Cosines.



$c^2 = a^2 + b^2 - 2ab \cos C$	Law of Cosines
$x^2 = 9^2 + 11^2 - 2(9)(11) \cos 28°$	Substitution
$x^2 = 202 - 198 \cos 28°$	Simplify.
$x = \sqrt{202 - 198 \cos 28°}$	Take the square root of each side.
$x \approx 5.2$	Use a calculator.

Watch Out!

Order of operations Remember to follow the order of operations when simplifying expressions. Multiplication or division must be performed before addition or subtraction. So, $202 - 198 \cos 28°$ *cannot* be simplified to $4 \cos 28°$.

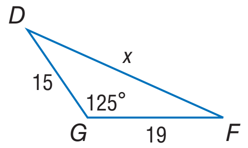
StudyTip

Obtuse Angles There are also values for sin *A*, cos *A*, and tan *A* when $A \geq 90°$. Values of the ratios for these angles can be found using the trigonometric functions on your calculator.

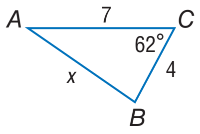
GuidedPractice

Find x. Round to the nearest tenth.

3A.



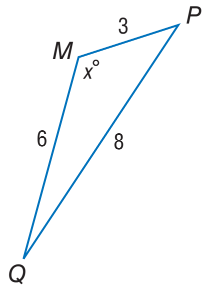
3B.



You can also use the Law of Cosines if you know three side measures (SSS).

Example 4 Law of Cosines (SSS)

Find x. Round to the nearest degree.



$m^2 = p^2 + q^2 - 2pq \cos M$ — Law of Cosines

$8^2 = 6^2 + 3^2 - 2(6)(3) \cos x°$ — Substitution

$64 = 45 - 36 \cos x°$ — Simplify.

$19 = -36 \cos x°$ — Subtract 45 from each side.

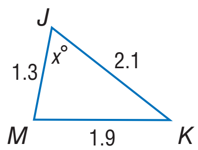
$\frac{19}{-36} = \cos x°$ — Divide each side by −36.

$x = \cos^{-1}\left(-\frac{19}{36}\right)$ — Use the inverse cosine ratio.

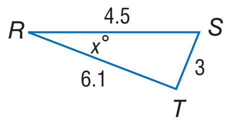
$x \approx 122$ — Use a calculator.

GuidedPractice

4A.



4B.



Real-WorldLink

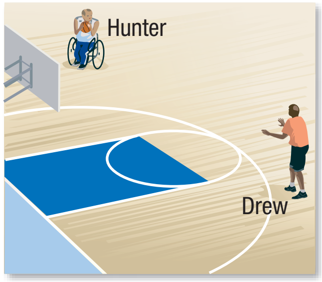
The first game of basketball was played at a YMCA in Springfield, Massachusetts, on December 1, 1891. James Naismith, a physical education instructor, invented the sport using a soccer ball and two half-bushel peach baskets, which is how the name *basketball* came about.

Source: *Encyclopaedia Britannica.*

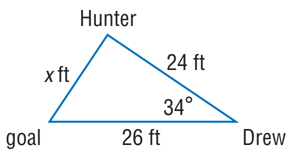
You can use the Law of Sines and Law of Cosines to solve direct and indirect measurement problems.

Real-World Example 5 Indirect Measurement

BASKETBALL **Drew and Hunter are playing basketball. Drew passes the ball to Hunter when he is 26 feet from the goal and 24 feet from Hunter. How far is Hunter from the goal if the angle from the goal to Drew and then to Hunter is 34°?**



Draw a diagram. Since we know two sides of a triangle and the included angle, use the Law of Cosines.



$x^2 = 24^2 + 26^2 - 2(24)(26) \cos 34°$ — Law of Cosines

$x = \sqrt{1252 - 1248 \cos 34°}$ — Simplify and take the positive square root of each side.

$x \approx 15$ — Use a calculator.

Hunter is about 15 feet from the goal when he takes his shot.

GuidedPractice

5. LANDSCAPING At 10 feet away from the base of a tree, the angle the top of a tree makes with the ground is 61°. If the tree grows at an angle of 78° with respect to the ground, how tall is the tree to the nearest foot?

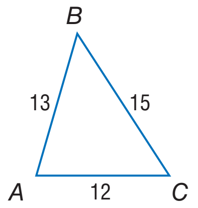
When solving right triangles, you can use sine, cosine, or tangent. When solving other triangles, you can use the Law of Sines or the Law of Cosines, depending on what information is given.

ReadingMath

Solve a Triangle Remember that to *solve* a triangle means to find all of the missing side measures and/or angle measures.

Example 6 Solve a Triangle

Solve triangle *ABC*. Round to the nearest degree.



Since $13^2 + 12^2 \neq 15^2$, this is not a right triangle. Since the measures of all three sides are given (SSS), begin by using the Law of Cosines to find $m\angle A$.

$a^2 = b^2 + c^2 - 2bc \cos A$	Law of Cosines
$15^2 = 12^2 + 13^2 - 2(12)(13) \cos A$	$a = 15$, $b = 12$, and $c = 13$
$225 = 313 - 312 \cos A$	Simplify.
$-88 = -312 \cos A$	Subtract 313 from each side.
$\frac{-88}{-312} = \cos A$	Divide each side by −312.
$m\angle A = \cos^{-1} \frac{88}{312}$	Use the inverse cosine ratio.
$m\angle A \approx 74$	Use a calculator.

Use the Law of Sines to find $m\angle B$.

$\frac{\sin A}{a} = \frac{\sin B}{b}$	Law of Sines
$\frac{\sin 74°}{15} \approx \frac{\sin B}{12}$	$m\angle A \approx 74$, $a = 15$, and $b = 12$
$12 \sin 74° = 15 \sin B$	Cross Products Property
$\frac{12 \sin 74°}{15} = \sin B$	Divide each side by 15.
$m\angle B = \sin^{-1} \frac{12 \sin 74°}{15}$	Use the inverse sine ratio.
$m\angle B \approx 50$	Use a calculator.

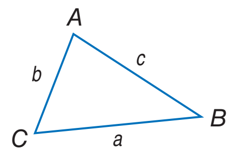
By the Triangle Angle Sum Theorem, $m\angle C \approx 180 - (74 + 50)$ or 56. Therefore $m\angle A \approx 74$, $m\angle B \approx 50$, and $m\angle C \approx 56$.

WatchOut

Rounding When you round a numerical solution and then use it in later calculations, your answers may be inaccurate. Wait until after you have completed all of your calculations to round.

GuidedPractice

Solve triangle *ABC* using the given information. Round angle measures to the nearest degree and side measures to the nearest tenth.



6A. $b = 10.2$, $c = 9.3$, $m\angle A = 26$

6B. $a = 6.4$, $m\angle B = 81$, $m\angle C = 46$

ConceptSummary Solving a Triangle

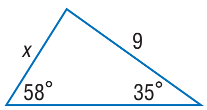
To solve . . .	Given	Begin by using . . .
a right triangle	leg-leg (LL) hypotenuse-leg (HL) acute angle-hypotenuse (AH) acute angle-leg (AL)	tangent ratio sine or cosine ratio sine or cosine ratio sine, cosine, or tangent ratios
any triangle	angle-angle-side (AAS) angle-side-angle (ASA) side-angle-side (SAS) side-side-side (SSS)	Law of Sines Law of Sines Law of Cosines Law of Cosines

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Examples 1–2 **Find x. Round angle measures to the nearest degree and side measures to the nearest tenth.**

1.



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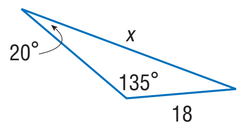

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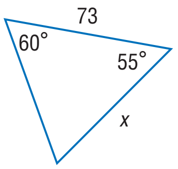

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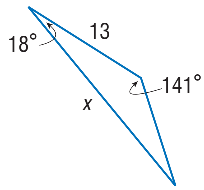
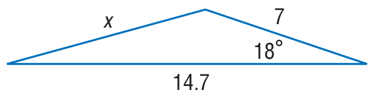


Examples 3–4 **5.**



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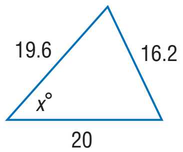
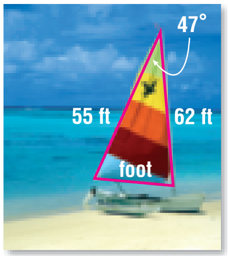


Example 5 **7. SAILING** Determine the length of the bottom edge, or foot, of the sail.



Example 6 **CCSS STRUCTURE** **Solve each triangle. Round angle measures to the nearest degree and side measures to the nearest tenth.**

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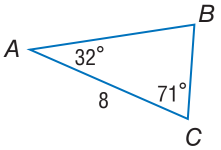

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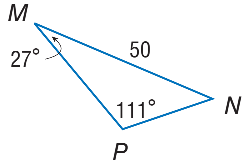

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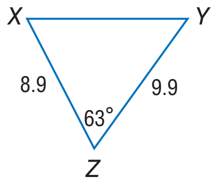
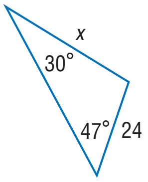


11. Solve $\triangle DEF$ if $DE = 16$, $EF = 21.6$, $FD = 20$.

Examples 1–2 Find x. Round side measures to the nearest tenth.

12.



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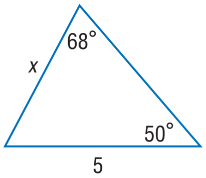

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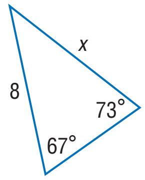

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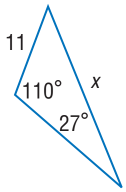

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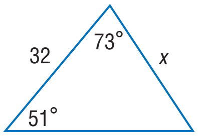

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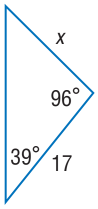

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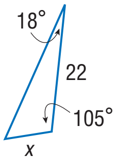

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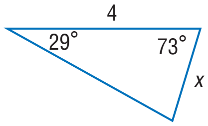

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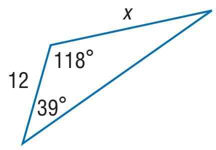
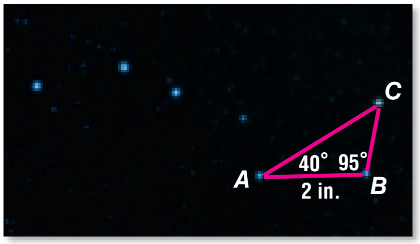


21. **CCSS MODELING** Angelina is looking at the Big Dipper through a telescope. From her view, the cup of the constellation forms a triangle that has measurements shown on the diagram at the right. Use the Law of Sines to determine distance between A and C.



Examples 3–4 Find x. Round angle measures to the nearest degree and side measures to the nearest tenth.

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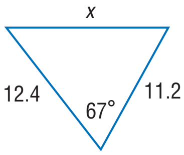

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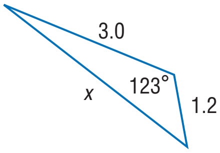

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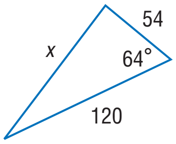

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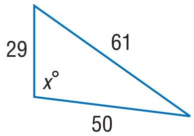

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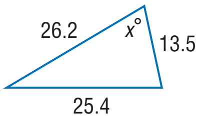

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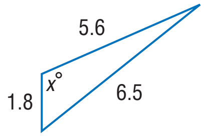
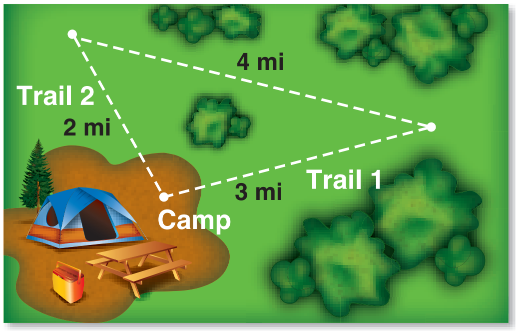
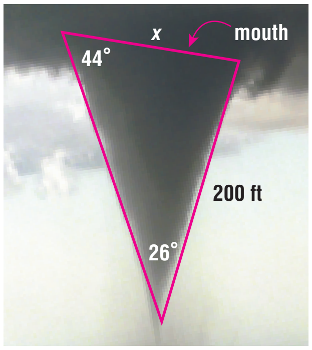


28. **HIKING** A group of friends who are camping decide to go on a hike. According to the map shown at the right, what is the measure of the angle between Trail 1 and Trail 2?

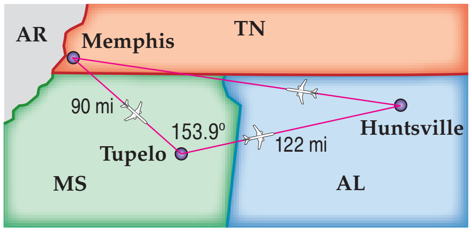


Example 5

29\. **TORNADOES** Find the width of the mouth of the tornado shown below.



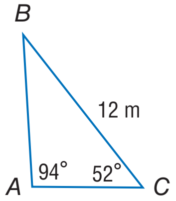
30\. **TRAVEL** A pilot flies 90 miles from Memphis, Tennessee, to Tupelo, Mississippi, to Huntsville, Alabama, and finally back to Memphis. How far is Memphis from Huntsville?



Example 6

CCSS STRUCTURE **Solve each triangle. Round angle measures to the nearest degree and side measures to the nearest tenth.**

31\.



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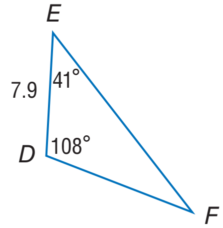

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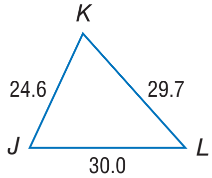

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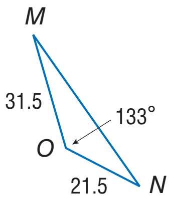

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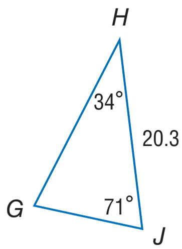

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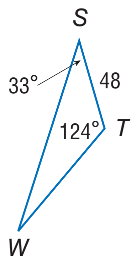

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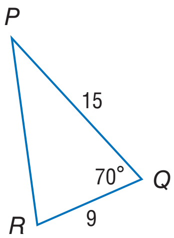

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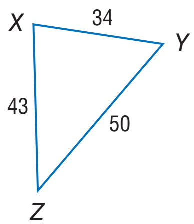
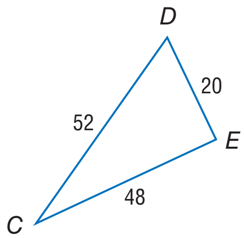


39\.

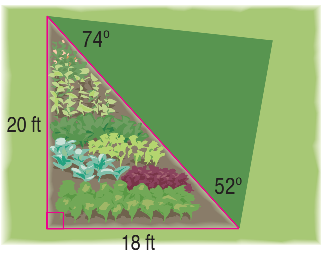


40\. Solve $\triangle JKL$ if $JK = 33$, $KL = 56$, $LJ = 65$.

41\. Solve $\triangle ABC$ if $m\angle B = 119$, $m\angle C = 26$, $CA = 15$.

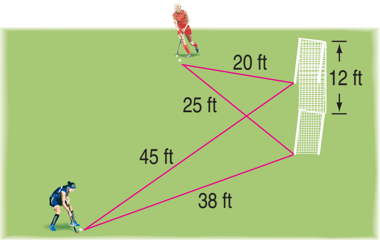
42\. Solve $\triangle XYZ$ if $XY = 190$, $YZ = 184$, $ZX = 75$.

43\. **GARDENING** Crystal has an organic vegetable garden. She wants to add another triangular section so that she can start growing tomatoes. If the garden and neighboring space have the dimensions shown, find the perimeter of the new garden to the nearest foot.



Jason Lafontaine/Painet Inc.

44. FIELD HOCKEY Alyssa and Nari are playing field hockey. Alyssa is standing 20 feet from one goal post and 25 feet from the opposite post. Nari is standing 45 feet from one goal post and 38 feet from the other post. If the goal is 12 feet wide, which player has a greater chance to make a shot? What is the measure of the player's angle?



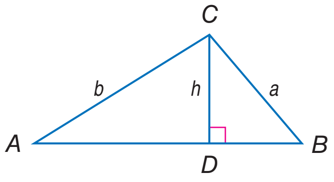
45. PROOF Justify each statement for the derivation of the Law of Sines.

Given: $\overline{CD}$ is an altitude of $\triangle ABC$.

Prove: $\frac{\sin A}{a} = \frac{\sin B}{b}$

Proof:

Statements	Reasons
$\overline{CD}$ is an altitude of $\triangle ABC$	Given
$\triangle ACD$ and $\triangle CBD$ are right	Def. of altitude
a. $\sin A = \frac{h}{b}$, $\sin B = \frac{h}{a}$	**a.** ___?___
b. $b \sin A = h$, $a \sin B = h$	**b.** ___?___
c. $b \sin A = a \sin B$	**c.** ___?___
d. $\frac{\sin A}{a} = \frac{\sin B}{b}$	**d.** ___?___



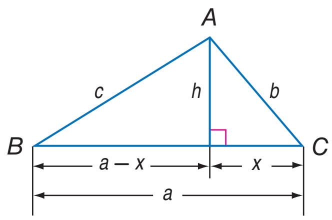
46. PROOF Justify each statement for the derivation of the Law of Cosines.

Given: h is an altitude of $\triangle ABC$.

Prove: $c^2 = a^2 + b^2 - 2ab \cos C$

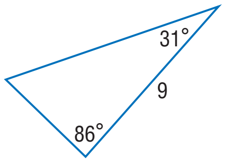
Proof:

Statements	Reasons
h is an altitude of $\triangle ABC$	Given
Altitude h separates $\triangle ABC$ into two right triangles	Def. of altitude
a. $c^2 = (a - x)^2 + h^2$	**a.** ___?___
b. $c^2 = a^2 - 2ax + x^2 + h^2$	**b.** ___?___
c. $x^2 + h^2 = b^2$	**c.** ___?___
d. $c^2 = a^2 - 2ax + b^2$	**d.** ___?___
e. $\cos C = \frac{x}{b}$	**e.** ___?___
f. $b \cos C = x$	**f.** ___?___
g. $c^2 = a^2 - 2a(b \cos C) + b^2$	**g.** ___?___
h. $c^2 = a^2 + b^2 - 2ab \cos C$	**h.** ___?___



CCSS SENSE-MAKING **Find the perimeter of each figure. Round to the nearest tenth.**

47.



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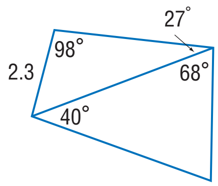

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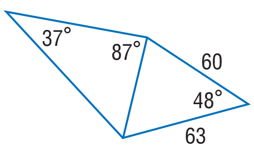

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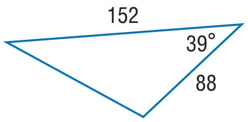
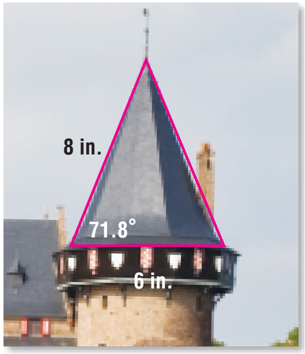


51 **MODELS** Vito is working on a model castle. Find the length of the missing side (in inches) using the diagram at the right.



52. **COORDINATE GEOMETRY** Find the measure of the largest angle in $\triangle ABC$ with coordinates $A(-3, 6)$, $B(4, 2)$, and $C(-5, 1)$. Explain your reasoning.

53. **MULTIPLE REPRESENTATIONS** In this problem, you will use trigonometry to find the area of a triangle.

a. **Geometric** Draw an acute, scalene $\triangle ABC$ including an altitude of length h originating at vertex A.

b. **Algebraic** Use trigonometry to represent h in terms of $m\angle B$.

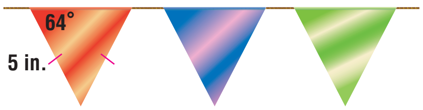
c. **Algebraic** Write an equation to find the area of $\triangle ABC$ using trigonometry.

d. **Numerical** If $m\angle B$ is 47, $AB = 11.1$, $BC = 14.1$, and $CA = 10.4$, find the area of $\triangle ABC$. Round to the nearest tenth.

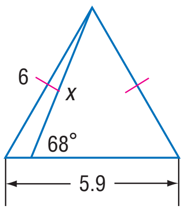
e. **Analytical** Write an equation to find the area of $\triangle ABC$ using trigonometry in terms of a different angle measure.

H.O.T. Problems Use Higher-Order Thinking Skills

54. **CCSS CRITIQUE** Colleen and Mike are planning a party. Colleen wants to sew triangular decorations and needs to know the perimeter of one of the triangles to buy enough trim. The triangles are isosceles with angle measurements of 64° at the base and side lengths of 5 inches. Colleen thinks the perimeter is 15.7 inches and Mike thinks it is 15 inches. Is either of them correct?



55. **CHALLENGE** Find the value of x in the figure at the right.



56. **REASONING** Explain why the Pythagorean Theorem is a specific case of the Law of Cosines.

57. **OPEN ENDED** Draw and label a triangle that can be solved:

a. using only the Law of Sines.

b. using only the Law of Cosines.

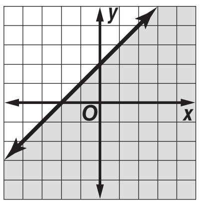
58. **WRITING IN MATH** What methods can you use to solve a triangle?

Standardized Test Practice

59. For $\triangle ABC$, $m\angle A = 42$, $m\angle B = 74$, and $a = 3$, what is the value of b?

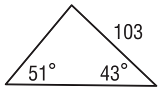
A 4.3 **C** 2.1
B 3.8 **D** 1.5

60. ALGEBRA Which inequality *best* describes the graph below?

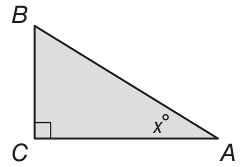


F $y \geq -x + 2$ **H** $y \geq -3x + 2$
G $y \leq x + 2$ **J** $y \leq 3x + 2$

61. SHORT RESPONSE What is the perimeter of the triangle shown below? Round to the nearest tenth.



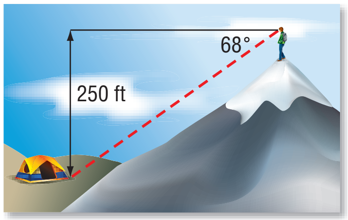
62. SAT/ACT If $\sin x = 0.6$ and $AB = 12$, what is the area of $\triangle ABC$?



A 9.6 units^2 **D** 34.6 units^2
B 28.8 units^2 **E** 42.3 units^2
C 31.2 units^2

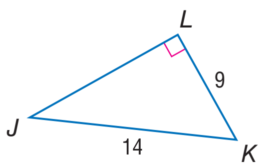
Spiral Review

63. HIKING A hiker is on top of a mountain 250 feet above sea level with a 68° angle of depression. She can see her camp from where she is standing. How far is her camp from the top of the mountain? (Lesson 8-5)

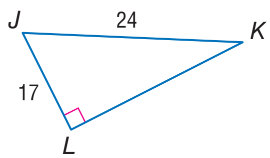


Use a calculator to find the measure of $\angle J$ to the nearest degree. (Lesson 8-4)

64.



65.



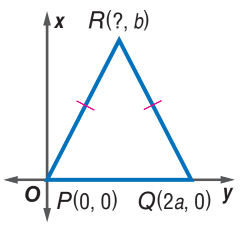
Determine whether the polygons are *always*, *sometimes*, or *never* similar. Explain your reasoning. (Lesson 7-2)

66. a right triangle and an isosceles triangle

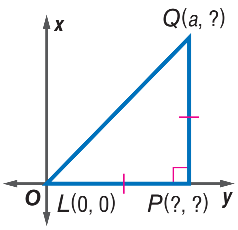
67. an equilateral triangle and a scalene triangle

Name the missing coordinates of each triangle. (Lesson 4-8)

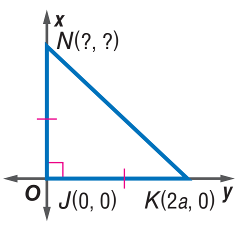
68.



69.



70.



Skills Review

Find the distance between each pair of points. Round to the nearest tenth.

71. $A(5, 1)$ and $C(-3, -3)$

72. $J(7, 11)$ and $K(-1, 5)$

73. $W(2, 0)$ and $X(8, 6)$