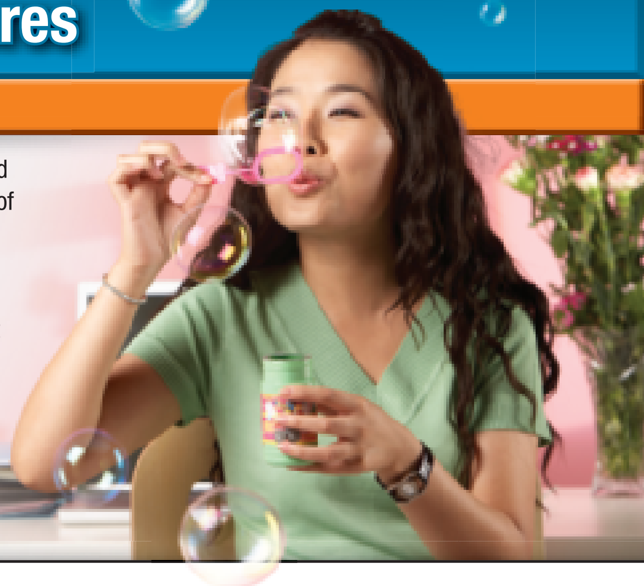
LESSON

12-6 Surface Areas and Volumes of Spheres



Then

- You found surface areas of prisms and cylinders.

Now

1. Find surface areas of spheres.
2. Find volumes of spheres.

Why?

- When you blow bubbles, soapy liquid surrounds a volume of air. Because of surface tension, the liquid maintains a shape that minimizes the surface area surrounding the air. The shape that minimizes surface area per unit of volume is a sphere.

NewVocabulary

great circle
pole
hemisphere

Common Core State Standards

Content Standards

G.GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.

G.GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★

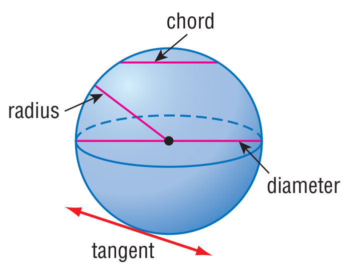
Mathematical Practices

1 Make sense of problems and persevere in solving them.
6 Attend to precision.

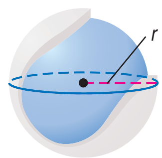
1 Surface Area of Spheres

Recall that a *sphere* is the locus of all points in space that are a given distance from a given point called the *center* of the sphere.

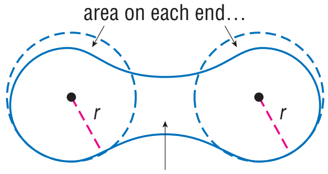
- A *radius* of a sphere is a segment from the center to a point on the sphere.
- A *chord* of a sphere is a segment that connects any two points on the sphere.
- A *diameter* of a sphere is a chord that contains the center.
- A *tangent* to a sphere is a line that intersects the sphere in exactly one point.



To develop a formula for the surface area of a sphere, consider a tennis ball. The covering of this sphere is comprised of two congruent dumbell-shaped pieces, each of which can be approximated by two congruent circles with radii equal to that of the sphere. So, the entire covering consists of approximately four congruent circles. The sum of these areas approximates the surface area of the sphere.



$S \approx 4A$	Sum of circles with area A
$\approx 4(\pi r^2)$ or $4\pi r^2$	$A = \pi r^2$



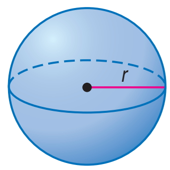
While its derivation is beyond the scope of this course, the exact formula is in fact $S = 4\pi r^2$.

KeyConcept Surface Area of a Sphere

Words The surface area S of a sphere is $S = 4\pi r^2$, where r is the radius.

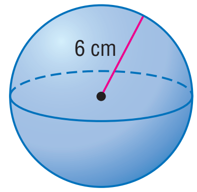
Symbols $S = 4\pi r^2$

Model



Example 1 Surface Area of a Sphere

Find the surface area of the sphere. Round to the nearest tenth.



$S = 4\pi r^2$ — Surface area of a sphere

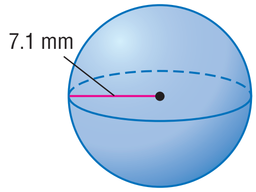
$= 4\pi(6)^2$ — Replace r with 6.

≈ 452.4 — Use a calculator.

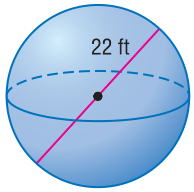
The surface area is about 452.4 square centimeters.

GuidedPractice

1A.



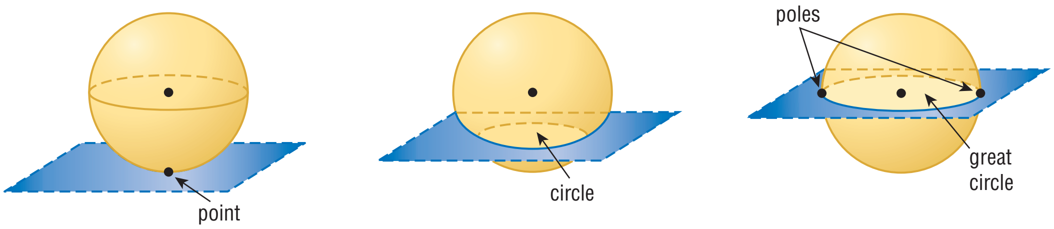
1B.



StudyTip

Great Circles A sphere has an infinite number of great circles.

A plane can intersect a sphere in a point or in a circle. If the circle contains the center of the sphere, the intersection is called a **great circle**. The endpoints of a diameter of a great circle are called **poles**.



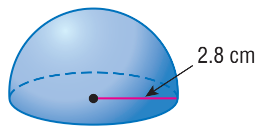
Since a great circle has the same center as the sphere and its radii are also radii of the sphere, it is the largest circle that can be drawn on a sphere. A great circle separates a sphere into two congruent halves, called **hemispheres**.

WatchOut!

Area of Hemisphere When finding the surface area of a hemisphere, do not forget to include the area of the great circle.

Example 2 Use Great Circles to Find Surface Area

a. Find the surface area of the hemisphere.



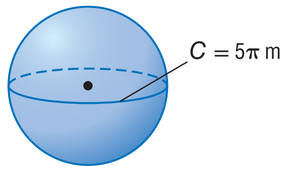
Find half the area of a sphere with a radius of 2.8 centimeters. Then add the area of the great circle.

$S = \frac{1}{2}(4\pi r^2) + \pi r^2$ — Surface area of a hemisphere

$= \frac{1}{2}[4\pi(2.8)^2] + \pi(2.8)^2$ — Replace r with 2.8.

$\approx 73.9 \text{ cm}^2$ — Use a calculator.

b. Find the surface area of a sphere if the circumference of the great circle is 5π meters.



First, find the radius. The circumference of a great circle is $2\pi r$. So, $2\pi r = 5\pi$ or $r = 2.5$.

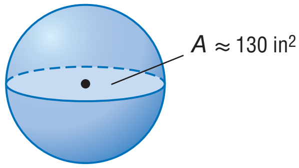
$S = 4\pi r^2$ — Surface area of a sphere

$= 4\pi(2.5)^2$ — Replace r with 2.5.

$\approx 78.5 \text{ m}^2$ — Use a calculator.

c. Find the surface area of a sphere if the area of the great circle is approximately 130 square inches.

First, find the radius. The area of a great circle is πr^2. So, $\pi r^2 = 130$ or $r \approx 6.4$.



$S = 4\pi r^2$	Surface area of a sphere
$\approx 4\pi(6.4)^2$ or about 514.7 in^2	Replace *r* with 6.4. Use a calculator.

GuidedPractice

Find the surface area of each figure. Round to the nearest tenth if necessary.

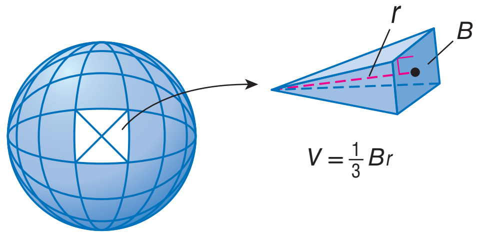
2A. sphere: circumference of great circle $= 16.2\pi$ ft

2B. hemisphere: area of great circle $\approx 94\ \text{mm}^2$

2C. hemisphere: circumference of great circle $= 36\pi$ cm

2 Volume of Spheres

Suppose a sphere with radius *r* contains infinitely many pyramids with vertices at the center of the sphere. Each pyramid has height *r* and base area *B*. The sum of the volumes of all the pyramids equals the volume of the sphere.



$V = \frac{1}{3}B_1r_1 + \frac{1}{3}B_2r_2 + \ldots + \frac{1}{3}B_nr_n$	Sum of volumes of pyramids
$= \frac{1}{3}r(B_1 + B_2 + \ldots + B_n)$	Distributive Property
$= \frac{1}{3}r(4\pi r^2)$	The sum of the pyramid base areas equals the surface area of the sphere.
$= \frac{4}{3}\pi r^3$	Simplify.

StudyTip

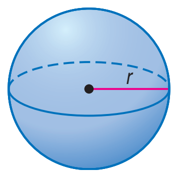
Draw a Diagram When solving problems involving volumes of solids, it is helpful to draw and label a diagram when no diagram is provided.

KeyConcept Volume of a Sphere

Words The volume V of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

Symbols $V = \frac{4}{3}\pi r^3$

Model



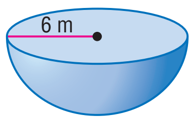
Example 3 Volumes of Spheres and Hemispheres

Find the volume of each sphere or hemisphere. Round to the nearest tenth.

a. a hemisphere with a radius of 6 meters

Estimate: $V \approx \frac{1}{2} \cdot \frac{4}{3} \cdot 3 \cdot 6^3$ or 432 m^3

$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$	Volume of a hemisphere
$= \frac{2}{3}\pi(6)^3$ or about 452.4 m^3	Replace *r* with 6. Use a calculator.



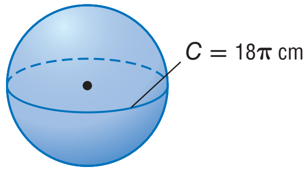
The volume of the hemisphere is about 452.4 cubic meters. This is close to the estimate, so the answer is reasonable.

StudyTip

CCSS Precision Remember to use the correct units when giving your answers. As with other solids, the surface area of a sphere is measured in square units, and volume is measured in cubic units.

b. a sphere with a great circle circumference of 18π centimeters

Step 1 Find the radius of the sphere.



$C = 2\pi r$ — Circumference of a circle

$18\pi = 2\pi r$ — Replace C with 18π.

$r = 9$ — Solve for r.

Step 2 Find the volume.

$V = \frac{4}{3}\pi r^3$ — Volume of a sphere

$= \frac{4}{3}\pi(9)^3$ or about 3053.6 cm^3 — Replace r with 9. Use a calculator.

GuidedPractice

3A. sphere: diameter = 7.4 in.

3B. hemisphere: area of great circle ≈ 249 mm^2

Real-WorldLink

The University of North Carolina has won the greatest number of national championships in women's soccer since the first tournament in 1982. As of 2009, they have won 18 times.

Source: Fact Monster

Real-World Example 4 Solve Problems Involving Solids

SOCCER The soccer ball globe at the right was constructed for the 2006 World Cup soccer tournament. It takes up 47,916π cubic feet of space. Assume that the globe is a sphere. What is the circumference of the globe?

Understand You know that the volume of the globe is 47,916π cubic feet. The circumference of the globe is the circumference of the great circle.

Plan First use the volume formula to find the radius. Then find the circumference of the great circle.

Solve

$V = \frac{4}{3}\pi r^3$ — Volume of a sphere

$47{,}916\pi = \frac{4}{3}\pi r^3$ — Replace V with $47{,}916\pi$.

$35{,}937 = r^3$ — Divide each side by $\frac{4}{3}\pi$.

Use a calculator to find $\sqrt[3]{35{,}937}$.

35937 [^] [(] 1 [÷] 3 [)] [ENTER] **33**

The radius of the globe is 33 feet. So, the circumference is $2\pi r = 2\pi(33)$ or approximately 207.3 feet.

Check You can work backward to check the solution.

If $C \approx 207.3$, then $r \approx 33$. If $r \approx 33$, then $V \approx \frac{4}{3}\pi \cdot 33^3$ or about 47,917π cubic feet. The solution is correct. ✓

GuidedPractice

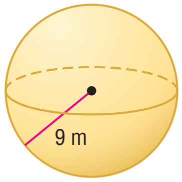
4. BALLOONS Ren inflates a spherical balloon to a circumference of about 14 inches. He then adds more air to the balloon until the circumference is about 18 inches. What volume of air was added to the balloon?

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

Examples 1–2 **Find the surface area of each sphere or hemisphere. Round to the nearest tenth.**

1.



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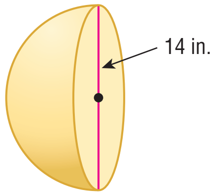


3. sphere: area of great circle $= 36\pi$ yd^2

4. hemisphere: circumference of great circle ≈ 26 cm

Example 3 **Find the volume of each sphere or hemisphere. Round to the nearest tenth.**

5. sphere: radius $= 10$ ft

6. hemisphere: diameter $= 16$ cm

7. hemisphere: circumference of great circle $= 24\pi$ m

8. sphere: area of great circle $= 55\pi$ in^2

Example 4

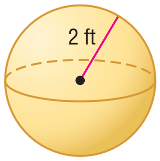
9. **BASKETBALL** Basketballs used in professional games must have a circumference of $29\frac{1}{2}$ inches. What is the surface area of a basketball used in a professional game?

Practice and Problem Solving

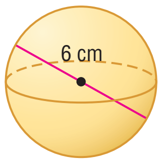
Extra Practice is on page R12.

Examples 1–2 **Find the surface area of each sphere or hemisphere. Round to the nearest tenth.**

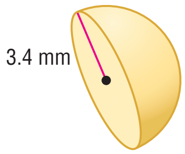
10.



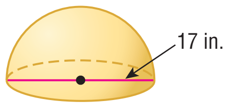
11.



12.



13.



14. sphere: circumference of great circle $= 2\pi$ cm

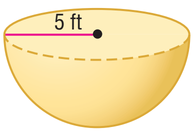
15. sphere: area of great circle ≈ 32 ft^2

16. hemisphere: area of great circle ≈ 40 in^2

17. hemisphere: circumference of great circle $= 15\pi$ mm

Example 3 CCSS **PRECISION** **Find the volume of each sphere or hemisphere. Round to the nearest tenth.**

18.



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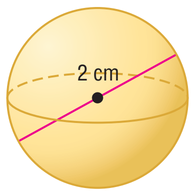


20. sphere: radius $= 1.4$ yd

21. hemisphere: diameter $= 21.8$ cm

22. sphere: area of great circle $= 49\pi$ m^2

23. sphere: circumference of great circle ≈ 22 in.

24. hemisphere: circumference of great circle ≈ 18 ft

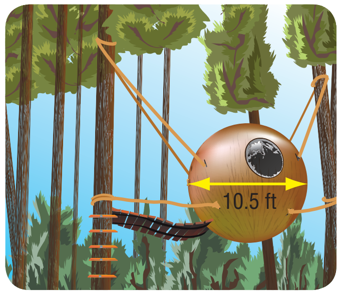
25. hemisphere: area of great circle ≈ 35 m^2

Example 4

26. FISH A *puffer fish* is able to "puff up" when threatened by gulping water and inflating its body. The puffer fish at the right is approximately a sphere with a diameter of 5 inches. Its surface area when inflated is about 1.5 times its normal surface area. What is the surface area of the fish when it is *not* puffed up?

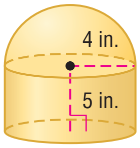
27. ARCHITECTURE The Reunion Tower in Dallas, Texas, is topped by a spherical dome that has a surface area of approximately 13,924π square feet. What is the volume of the dome? Round to the nearest tenth.

28. TREE HOUSE The spherical tree house, or *tree sphere,* shown at the right has a diameter of 10.5 feet. Its volume is 1.8 times the volume of the first tree sphere that was built. What was the diameter of the first tree sphere? Round to the nearest foot.

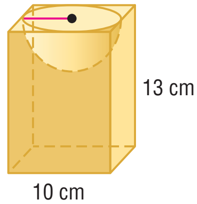


CCSS SENSE-MAKING **Find the surface area and the volume of each solid. Round to the nearest tenth.**

29.



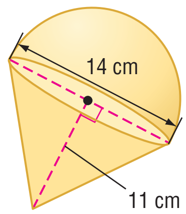
30.



31. TOYS The spinning top at the right is a composite of a cone and a hemisphere.

a. Find the surface area and the volume of the top. Round to the nearest tenth.

b. If the manufacturer of the top makes another model with dimensions that are one-half of the dimensions of this top, what are its surface area and volume?



32. BALLOONS A spherical helium-filled balloon with a diameter of 30 centimeters can lift a 14-gram object. Find the size of a balloon that could lift a person who weighs 65 kilograms. Round to the nearest tenth.

Use sphere *S* to name each of the following.

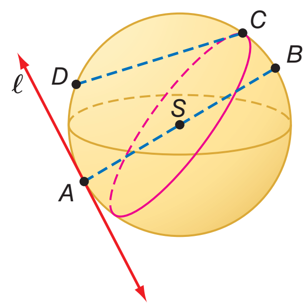
33. a chord

34. a radius

35. a diameter

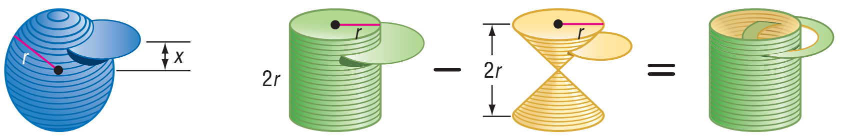
36. a tangent

37. a great circle



38. DIMENSIONAL ANALYSIS Which has greater volume: a sphere with a radius of 2.3 yards or a cylinder with a radius of 4 feet and height of 8 feet?

39. **INFORMAL PROOF** A sphere with radius r can be thought of as being made up of a large number of discs or thin cylinders. Consider the disc shown that is x units above or below the center of the sphere. Also consider a cylinder with radius r and height $2r$ that is hollowed out by two cones of height and radius r.



a. Find the radius of the disc from the sphere in terms of its distance x above the sphere's center. (*Hint:* Use the Pythagorean Theorem.)

b. If the disc from the sphere has a thickness of y units, find its volume in terms of x and y.

c. Show that this volume is the same as that of the hollowed-out disc with thickness of y units that is x units above the center of the cylinder and cone.

d. Since the expressions for the discs at the same height are the same, what guarantees that the hollowed-out cylinder and sphere have the same volume?

e. Use the formulas for the volumes of a cylinder and a cone to derive the formula for the volume of the hollowed-out cylinder and thus, the sphere.

CCSS TOOLS Describe the number and types of planes that produce reflection symmetry in each solid. Then describe the angles of rotation that produce rotation symmetry in each solid.

40. sphere

41. hemisphere

CHANGING DIMENSIONS A sphere has a radius of 12 centimeters. Describe how each change affects the surface area and the volume of the sphere.

42. The radius is multiplied by 4.

43. The radius is divided by 3.

44. **DESIGN** A standard juice box holds 8 fluid ounces.

a. Sketch designs for three different juice containers that will each hold 8 fluid ounces. Label dimensions in centimeters. At least one container should be cylindrical. (*Hint:* $1 \text{ fl oz} \approx 29.57353 \text{ cm}^3$)

b. For each container in part **a**, calculate the surface area to volume (cm^2 per fl oz) ratio. Use these ratios to decide which of your containers can be made for the lowest materials cost. What shape container would minimize this ratio, and would this container be the cheapest to produce? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

45. **CHALLENGE** A cube has a volume of 216 cubic inches. Find the volume of a sphere that is circumscribed about the cube. Round to the nearest tenth.

46. **REASONING** Determine whether the following statement is *true* or *false*. If true, explain your reasoning. If false, provide a counterexample.

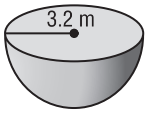
If a sphere has radius r, there exists a cone with radius r having the same volume.

47. **OPEN ENDED** Sketch a sphere showing two examples of great circles. Sketch another sphere showing two examples of circles formed by planes intersecting the sphere that are *not* great circles.

48. **WRITING IN MATH** Write a ratio comparing the volume of a sphere with radius r to the volume of a cylinder with radius r and height $2r$. Then describe what the ratio means.

Standardized Test Practice

49. GRIDDED RESPONSE What is the volume of the hemisphere shown below in cubic meters?



50. ALGEBRA What is the solution set of $3z + 4 < 6 + 7z$?

A $\{z \mid z > -0.5\}$ **C** $\{z \mid z < -0.5\}$
B $\{z \mid z > -2\}$ **D** $\{z \mid z < -2\}$

51. If the area of the great circle of a sphere is 33 ft^2, what is the surface area of the sphere?

F 42 ft^2 **H** 132 ft^2
G 117 ft^2 **J** 264 ft^2

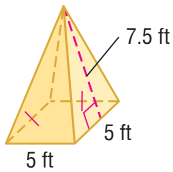
52. SAT/ACT If a line ℓ is a perpendicular bisector of segment *AB* at *E*, how many points on line ℓ are the same distance from point *A* as from point *B*?

A none **D** three
B one **E** all points
C two

Spiral Review

Find the volume of each pyramid. Round to the nearest tenth if necessary. (Lesson 12-5)

53.



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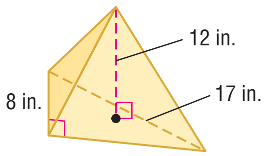

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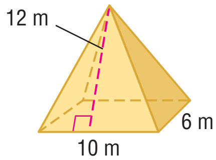
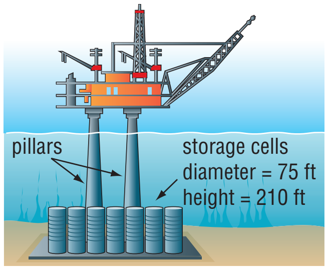
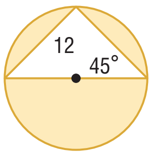


56. ENGINEERING The base of an oil drilling platform is made up of 24 concrete cylindrical cells. Twenty of the cells are used for oil storage. The pillars that support the platform deck rest on the four other cells. Find the total volume of the storage cells. (Lesson 12-4)

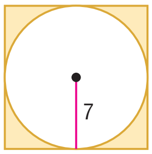


Find the area of each shaded region. Round to the nearest tenth. (Lesson 11-4)

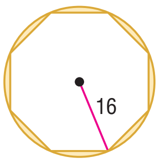
57.



58.



59.



COORDINATE GEOMETRY Find the area of each figure. (Lesson 11-1)

60. ▱*WXYZ* with $W(0, 0)$, $X(4, 0)$, $Y(5, 5)$, and $Z(1, 5)$

61. $\triangle ABC$ with $A(2, -3)$, $B(-5, -3)$, and $C(-1, 3)$

Skills Review

Refer to the figure.

62. How many planes appear in this figure?

63. Name three points that are collinear.

64. Are points *G*, *A*, *B*, and *E* coplanar? Explain.

65. At what point do $\overleftrightarrow{EF}$ and $\overleftrightarrow{AB}$ intersect?